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Skew-orthogonal polynomials, differential systems and random matrix theory

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Abstract
We study skew-orthogonal polynomials with respect to the weight function $\exp[-2V(x)]$, with $V(x) = \sum_{K=1}^{2d}(u_K/K)x^K, u_{2d} > 0, d > 0$. A finite subsequence of such skew-orthogonal polynomials arising in the study of orthogonal and symplectic ensembles of random matrices satisfies a system of differential-difference-deformation equation. The vectors formed by such subsequence have the rank equal to the degree of the potential in the quaternion sense. These solutions satisfy certain compatibility condition and hence admit a simultaneous fundamental system of solutions.

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1. Introduction

1.1. Random matrices

The concept of 'universality' in random matrix theory and its various applications in real physical systems have attracted both mathematicians and physicists in the last few decades [1–14]. From mathematical point of view, the study of 'universality' in the energy level correlations of random matrices requires a good understanding of the asymptotic behaviour of certain families of polynomials. For example, unitary ensembles of random matrices require an understanding of the large n behaviour of orthogonal (the one-matrix model) and bi-orthogonal (the two-matrix model) polynomials, while orthogonal and symplectic ensembles require that of the skew-orthogonal polynomials. The rich literature available on orthogonal polynomials [6–18] and bi-orthogonal polynomials [19–23] has contributed a lot in our understanding of the unitary ensembles. Our aim is to develop the theory of skew-orthogonal polynomials [3–5, 24–27] so that we can have further insight into the one-matrix model for orthogonal and symplectic ensembles of random matrices.

In this direction, previous experience with orthogonal and bi-orthogonal polynomials arising in the two-matrix model makes us believe that perhaps the most logical and rigorous way to study the asymptotic properties of these skew-orthogonal polynomials is to do the following:

(1) first, we must look for a finite subsequence of skew-orthonormal vectors which will satisfy a mutually compatible system of differential-difference-deformation equation. This requires one to fix the 'size' or rank of this finite subsequence. The generalized Christoffel–Darboux formula helps in this regard as it gives an indication of the number of terms near the so-called 'Fermi-level' which are actually required to study this system. (2) The next step is to look for a mutually compatible system of differential-difference-deformation equation satisfied by the finite subsequence of these vectors and hence find the so-called 'fundamental solution'. (3) Finally, to formulate a quaternion matrix Riemann–Hilbert problem and by applying the steepest descent method, obtain the Plancherel–Rotach-type [15] formula for skew-orthogonal polynomials.

Reference [5] achieved the first goal. In this paper, we try to address the second issue of the existence of the 'fundamental solution' of a system of differential-difference-deformation equation. One of the by-products is that we observe certain duality property between the skew-orthonormal vectors of orthogonal and symplectic ensembles. (In fact, this justifies further the use of skew-orthogonal polynomials to ordinary orthogonal polynomials in studying these ensembles [6, 7, 28–36].) The third part, which is also the most crucial one, is not yet fully understood.

We shall consider an ensemble of $2N$-dimensional matrices H with probability distribution

$$P_{\beta,N}(H)\,\mathrm{d}H = \frac{1}{\mathcal{Z}_{\beta N}} \exp[-[2\,\mathrm{Tr}\,V(H)]]\,\mathrm{d}H, \tag{1.1}$$

where the parameter $\beta = 1$ and 4 corresponds to the joint probability density for orthogonal and symplectic ensembles of random matrices. The 'potential' $V(x)$ is a polynomial of degree $2d$ with positive leading coefficient:

$$V(x) = \sum_{K=1}^{2d} \frac{u_K}{K} x^K, \qquad u_{2d} > 0, \qquad d > 0, \tag{1.2}$$

where u_K is called the deformation parameter. $\mathcal{Z}_{\beta N}$ is the so-called 'partition function':

$$\mathcal{Z}_{\beta N} := \int_{H\in M_{2N}^{(\beta)}} \exp[-[2\,\mathrm{Tr}\,V(H)]]\,\mathrm{d}H = N! \prod_{j=0}^{N-1} g_j^{(\beta)}, \tag{1.3}$$

where $M_{2N}^{(\beta)}$ is a set of all $2N \times 2N$ real symmetric ($\beta = 1$) and quaternion real self-dual ($\beta = 4$) matrices. $\mathrm{d}H$ is the standard Haar measure. $g_j^{(\beta)}$ is the skew-normalization constant for polynomials related to orthogonal and symplectic ensembles [24].

Remark on notation. Before entering the details of calculation, we must mention that throughout this paper we have followed to a great extent notation used in [17, 19–21, 25]. Apart from having the advantage of using a well-established and compact notation, this strategy will also highlight the striking similarity (as well as the difference) between skew-orthogonal polynomials and their bi-orthogonal counterpart.

1.2. Skew-orthogonal polynomials: relevance in orthogonal and symplectic ensembles

Definition. For orthogonal and symplectic ensembles of random matrices, we define semi-infinite vectors:

$$\Phi^{(\beta)}(x) = \left(\Phi_0^{(\beta)^t}(x) \cdots \Phi_n^{(\beta)^t}(x) \cdots\right)^t, \qquad \widehat{\Phi}^{(\beta)}(x) = -\Phi^{(\beta)^t}(x)Z, \tag{1.4}$$

$$\Psi^{(\beta)}(x) = \left(\Psi_0^{(\beta)^t}(x) \cdots \Psi_n^{(\beta)^t}(x) \cdots\right)^t, \qquad \widehat{\Psi}^{(\beta)}(x) = -\Psi^{(\beta)^t}(x)Z, \tag{1.5}$$

where

$$\Psi_n^{(4)}(x) = \Phi_n'^{(4)}(x), \qquad \Psi_n^{(1)}(x) = \int_{\mathbb{R}} \Phi_n^{(1)}(y)\epsilon(x-y)\,\mathrm{d}y. \tag{1.6}$$

Each entry in these semi-infinite vectors is a (2×1) matrix:

$$\Phi_n^{(\beta)}(x) = \begin{pmatrix} \phi_{2n}^{(\beta)}(x) \\ \phi_{2n+1}^{(\beta)}(x) \end{pmatrix}, \qquad \Psi_n^{(\beta)}(x) = \begin{pmatrix} \psi_{2n}^{(\beta)}(x) \\ \psi_{2n+1}^{(\beta)}(x) \end{pmatrix}, \tag{1.7}$$

where

$$\phi_n^{(\beta)}(x) = \frac{1}{\sqrt{g_n^{(\beta)}}} \pi_n^{(\beta)}(x) \exp[-V(x)] \tag{1.8}$$

is the weighted skew-normalized polynomial (often called the quasi-polynomial) and

$$\pi_n^{(\beta)}(x) = \sum_{k=0}^{n} c_k^{(n,\beta)} x^k, \qquad c_n^{(n,\beta)} = 1, \qquad n \in \mathbb{N} \tag{1.9}$$

is monic skew-orthogonal polynomial of order n.

$$Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dotplus \cdots \dotplus \tag{1.10}$$

is semi-infinite anti-symmetric block-diagonal matrix with $Z^2 = -1$ and

$$\epsilon(r) = \frac{|r|}{r} \tag{1.11}$$

is the step function. These vectors satisfy skew-orthonormality relation:

$$\left(\Phi_n^{(\beta)}, \widehat{\Psi}_m^{(\beta)}\right) \equiv \int_{\mathbb{R}} \Phi_n^{(\beta)}(x)\widehat{\Psi}_m^{(\beta)}(x)\,\mathrm{d}x = \delta_{nm}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad n, m \in \mathbb{N}. \tag{1.12}$$

Remark. This definition differs from that of [3, 4, 26] for $\beta = 4$ by a factor of 2, which is incorporated in the normalization constant.

These skew-orthonormal vectors form the necessary constituents of the kernel functions required to study different statistical properties of orthogonal ensembles and symplectic ensembles of random matrices. For example, the two-point correlation function can be written in terms of the 2×2 kernel function [26, 27]:

$$\sigma_2^{(\beta)}(x, y) = \begin{pmatrix} S_{2N}^{(\beta)}(x, y) & D_{2N}^{(\beta)}(x, y) \\ I_{2N}^{(\beta)}(x, y) + \frac{\delta_{1,\beta}}{2}\epsilon(x-y) & S_{2N}^{(\beta)}(y, x) \end{pmatrix}, \tag{1.13}$$

where δ is the Kronecker delta and

$$S_{2N}^{(\beta)}(x, y) = \widehat{\Psi}^{(\beta)}(x) \prod_N \Phi^{(\beta)}(y), \qquad D_{2N}^{(\beta)}(x, y) = \widehat{\Phi}^{(\beta)}(x) \prod_N \Phi^{(\beta)}(y), \tag{1.14}$$

$$I_{2N}^{(\beta)}(x, y) = \widehat{\Psi}^{(\beta)}(x) \prod_N \Psi^{(\beta)}(y), \qquad S_{2N}^{(\beta)}(y, x) = \widehat{\Phi}^{(\beta)}(x) \prod_N \Psi^{(\beta)}(y). \tag{1.15}$$

The level density is given by

$$\rho^{(\beta)}(x, x) = S_{2N}^{(\beta)}(x, x). \tag{1.16}$$

The matrix

$$\prod_N = \mathrm{diag}(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{N}, 0, \ldots, 0) \tag{1.17}$$

is formed by N (2×2) unit matrices (i.e., a unit matrix of size $2N$ in real space). In general, this will be used to truncate semi-infinite matrices and, in the above case, the semi-infinite vectors of the finite sum in equations (1.14) and (1.15).

1.3. The generalized Christoffel–Darboux sum

Here, we present a summary of the main results of [5], where we studied the kernel function $S_{2N}^{(\beta)}(x, y)$. We expand $(x\Phi^{(\beta)}(x))$, $(\Phi^{(\beta)}(x))'$ and $(x\Phi^{(\beta)}(x))'$ in terms of $\Phi^{(\beta)}(x)$ (and hence introduce the semi-infinite matrices $Q^{(\beta)}$, $P^{(\beta)}$ and $R^{(\beta)}$, respectively):

$$x\Phi^{(\beta)}(x) = Q^{(\beta)}\Phi^{(\beta)}(x), \tag{1.18}$$

$$\Psi^{(4)}(x) = P^{(4)}\Phi^{(4)}(x), \qquad x\Psi^{(4)}(x) = R^{(4)}\Phi^{(4)}(x), \tag{1.19}$$

$$2\Phi^{(1)}(x) = P^{(1)}\Psi^{(1)}(x), \qquad 2x\Phi^{(1)}(x) = R^{(1)}\Psi^{(1)}(x), \tag{1.20}$$

where equation (1.20) is obtained by multiplying the above expansion by $\epsilon(y - x)$ and integrating by parts. Here,

$$R^{(4)} = P^{(4)}Q^{(4)}, \qquad R^{(1)} = Q^{(1)}P^{(1)}. \tag{1.21}$$

The matrices $P^{(\beta)}$, $Q^{(\beta)}$ and $R^{(\beta)}$ are quaternion matrices. For a nice introduction to the subject, the reader is referred to the book by Professor M L Mehta [27].

Defining the quaternion matrix

$$\overline{R}^{(4)}(x) := R^{(4)} - xP^{(4)}, \qquad \overline{R}^{(1)}(x) := \tfrac{1}{2}[R^{(1)} - xP^{(1)}], \tag{1.22}$$

the generalized Christoffel–Darboux formula for the symplectic ensembles ($\beta = 4$) can be written as

$$(x - y)S_{2N}^{(4)}(x, y) = -\widehat{\Phi}^{(4)}(x)\left[\overline{R}^{(4)}(x), \prod_N\right]\Phi^{(4)}(y). \tag{1.23}$$

For orthogonal ensembles ($\beta = 1$), the generalized Christoffel–Darboux is given by

$$(y - x)S_{2N}^{(1)}(x, y) = -\widehat{\Psi}^{(1)}(x)\left[\overline{R}^{(1)}(y), \prod_N\right]\Psi^{(1)}(y). \tag{1.24}$$

The generalized Christoffel–Darboux matrix $\left[\overline{R}^{(\beta)}(x), \prod_N\right]$ in terms of the elements of the quaternion matrix takes the form

$$\left[\overline{R}^{(\beta)}(x), \prod_N\right] = \begin{pmatrix} 0 & 0 & 0 & \overline{R}^{(\beta)}_{2N-2d,2N}(x) & 0 & 0 \\ 0 & 0 & 0 & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \overline{R}^{(\beta)}_{2N-1,2N}(x) & \cdots & \overline{R}^{(\beta)}_{2N-1,2N+2d-1}(x) \\ -\overline{R}^{(\beta)}_{2N,2N-2d}(x) & \cdots & -\overline{R}^{(\beta)}_{2N,2N-1}(x) & 0 & 0 & 0 \\ 0 & \ddots & \vdots & \vdots & 0 & 0 \\ 0 & 0 & -\overline{R}^{(\beta)}_{2N+2d-1,2N-1}(x) & 0 & \cdots & 0 \end{pmatrix}, \qquad N \geqslant d. \tag{1.25}$$

Here, we must point out that there is a small difference with the notation used in the second part of [5], where we have used a matrix of size $2N + 2$ to prove the 'universality' in the Gaussian case. Also the definition of $R^{(1)}$ and $P^{(1)}$ differs by a factor of 2.

1.4. Difference-differential-deformation equations

A quick glance at equation (1.25) reveals that the relevant vectors contributing to the correlation function are $\phi^{(\beta)}_{(2N-2d)}(x)\cdots\phi^{(\beta)}_{(2N+2d-1)}(x)$. This prompts us to define a finite subsequence (or window: $W \Rrightarrow \{N-d,\ldots,N+d-1\}$, $N \geqslant d$) of skew-orthogonal vectors of size $2d$ (in the quaternionic sense), $\Phi^{(\beta)}_W(x)$ and $\Psi^{(\beta)}_W(x)$:

$$\Phi^{(\beta)}_W(x) := \begin{pmatrix} \phi^{(\beta)}_{2N-2d}(x) \\ \vdots \\ \phi^{(\beta)}_{2N+2d-1}(x) \end{pmatrix} \equiv \begin{pmatrix} \Phi^{(\beta)}_{N-d}(x) \\ \vdots \\ \Phi^{(\beta)}_{N+d-1}(x) \end{pmatrix}, \qquad N \geqslant d, \tag{1.26}$$

$$\Psi^{(\beta)}_W(x) := \begin{pmatrix} \psi^{(\beta)}_{2N-2d}(x) \\ \vdots \\ \psi^{(\beta)}_{2N+2d-1}(x) \end{pmatrix} \equiv \begin{pmatrix} \Psi^{(\beta)}_{N-d}(x) \\ \vdots \\ \Psi^{(\beta)}_{N+d-1}(x) \end{pmatrix}, \qquad N \geqslant d, \tag{1.27}$$

such that

$$\Phi^{(\beta)}_{W\pm j}(x) := \left(\Phi^{(\beta)^t}_{N-d\pm j}(x)\cdots\Phi^{(\beta)^t}_{N+d-1\pm j}(x)\right)^t, \qquad \forall j \in \mathbb{N}, \tag{1.28}$$

$$\Psi^{(\beta)}_{W\pm j}(x) := \left(\Psi^{(\beta)^t}_{N-d\pm j}(x)\cdots\Psi^{(\beta)^t}_{N+d-1\pm j}(x)\right)^t, \qquad \forall j \in \mathbb{N}. \tag{1.29}$$

The rank of the window is equal to the degree of the potential $V(x)$ in the quaternionic space and twice that of the degree of $V(x)$ in the real space.

The recursion relations connecting the finite subsequence (or window) with the upper or lower one are through the ladder operator

$$\Phi^{(4)}_{W+1}(x) = A^{(4)}_N(x)\Phi^{(4)}_W(x), \qquad \Phi^{(4)}_{W-1}(x) = \left(A^{(4)}_{N-1}(x)\right)^{-1}\Phi^{(4)}_W(x), \tag{1.30}$$

$$\Psi^{(1)}_{W+1}(x) = A^{(1)}_N(x)\Psi^{(1)}_W(x), \qquad \Psi^{(1)}_{W-1}(x) = \left(A^{(1)}_{N-1}(x)\right)^{-1}\Psi^{(1)}_W(x), \tag{1.31}$$

and

$$\Psi^{(4)}_{W+1}(x) = B^{(4)}_N(x)\Psi^{(4)}_W(x), \qquad \Psi^{(4)}_{W-1}(x) = \left(B^{(4)}_{N-1}(x)\right)^{-1}\Psi^{(4)}_W(x), \tag{1.32}$$

$$\Phi^{(1)}_{W+1}(x) = B^{(1)}_N(x)\Phi^{(1)}_W(x), \qquad \Phi^{(1)}_{W-1}(x) = \left(B^{(1)}_{N-1}(x)\right)^{-1}\Phi^{(1)}_W(x). \tag{1.33}$$

The vectors $\Phi^{(\beta)}_W(x)$ and $\Psi^{(\beta)}_W(x)$ also satisfy a system of ODEs:

$$\begin{array}{l|l} \dfrac{\mathrm{d}}{\mathrm{d}x}\Psi^{(4)}_W(x) = \underline{D}^{(4)}_N(x)\Psi^{(4)}_W(x), & \dfrac{\mathrm{d}}{\mathrm{d}x}\Phi^{(1)}_W(x) = \underline{D}^{(1)}_N(x)\Phi^{(1)}_W(x), \\ \hline \dfrac{\mathrm{d}}{\mathrm{d}x}\Phi^{(4)}_W(x) = D^{(4)}_N(x)\Phi^{(4)}_W(x), & \dfrac{\mathrm{d}}{\mathrm{d}x}\Psi^{(1)}_W(x) = D^{(1)}_N(x)\Psi^{(1)}_W(x). \end{array} \tag{1.34}$$

Under an infinitesimal change of the deformation parameter u_K (the coefficients of the polynomial potential $V(x)$), these vectors satisfy a system of PDEs given by

$$\begin{array}{l|l} \dfrac{\partial}{\partial u_K}\Psi^{(4)}_W(x) = \underline{U}^{N^{(4)}}_K(x)\Psi^{(4)}_W(x), & \dfrac{\partial}{\partial u_K}\Phi^{(1)}_W(x) = \underline{U}^{N^{(1)}}_K(x)\Phi^{(1)}_W(x), \\ \hline \dfrac{\partial}{\partial u_K}\Phi^{(4)}_W(x) = U^{N^{(4)}}_K(x)\Phi^{(4)}_W(x), & \dfrac{\partial}{\partial u_K}\Psi^{(1)}_W(x) = U^{N^{(1)}}_K(x)\Psi^{(1)}_W(x). \end{array} \tag{1.35}$$

Thus, the finite subsequence of skew-orthogonal vectors satisfies a system of difference-differential-deformation equation.

Remark. The two pairs of matrices $\underline{D}_N^{(4)}(x)$ and $\underline{D}_N^{(1)}(x)$ (similarly $D_N^{(1)}(x)$ and $D_N^{(4)}(x)$), $\underline{U}_K^{N\,(4)}(x)$ and $\underline{U}_K^{N\,(1)}(x)$ (similarly $U_K^{N\,(1)}(x)$ and $U_K^{N\,(4)}(x)$) and $A_N^{(4)}(x)$ and $A_N^{(1)}(x)$ (similarly $B_N^{(4)}(x)$ and $B_N^{(1)}(x)$) are dual in the sense that they remain invariant under an interchange of $\Psi^{(4)}(x) \mapsto \Phi^{(1)}(x)$ and $Q^{(4)} \mapsto Q^{(1)}$ (and similarly for $\Psi^{(1)}(x)$ and $\Phi^{(4)}(x)$).

1.5. Compatibility

The existence of recursion relations, differential equations and deformation equations for vectors arising in the orthogonal ensembles and symplectic ensembles of random matrices can be viewed as just a projection of the semi-infinite functions $\Phi^{(\beta)}(x)$ and $\Psi^{(\beta)}(x)$ onto the finite 'window' $\Phi_W^{(\beta)}(x)$ and $\Psi_W^{(\beta)}(x)$, respectively. However, we may also consider these equations as defining an overdetermined system of finite difference-differential-deformation equations of the vector functions and see that these systems are compatible. This leads to the existence of a fundamental matrix solution, denoted by $\Phi_{[W]}^{(\beta)}(x)$ and $\Psi_{[W]}^{(\beta)}(x)$ where all the column vectors satisfy the above difference-differential-deformation equations simultaneously.

The compatibility of the deformation and difference equation with the differential equation implies that the generalized monodromy of the operators $\left(\frac{\mathrm{d}}{\mathrm{d}x} - D_N^{(\beta)}(x)\right)$ and $\left(\frac{\mathrm{d}}{\mathrm{d}x} - \underline{D}_N^{(\beta)}(x)\right)$ is invariant under u_K deformations and shifts in N.

Outline of the paper. Section 2 deals with the properties of different finite band matrices related to skew-orthogonal polynomials. In section 3, we study the system of PDEs arising from the infinitesimal change of the deformation parameter u_K. We derive the difference relations satisfied by the finite subsequence of skew-orthogonal vectors in section 4. In section 5, we derive the folding function which is used to project any given vector $\Phi_n^{(\beta)}$ (or $\Psi_n^{(\beta)}$) onto its finite subsequence or window. In section 6, we obtain the folded deformation matrix, using the results of section 5. The differential equation for skew-orthonormal vectors, using the results obtained in section 6, is derived in section 7. In section 8, we discuss the existence of the Cauchy-like transforms of the skew-orthogonal vectors of order n as the other solutions to the differential-deformation-difference equations, for fairly large n. We prove compatibility conditions for these difference-deformation-differential systems in section 9.

2. Recursion relations and finite band matrices

In this section, we will study in detail the different recursion relations satisfied by the skew-orthonormal vectors. Here, we must note that unlike the orthogonal polynomials skew-orthonormal vectors do not satisfy a three-term recursion relation and hence do not give birth to the tri-diagonal Jacobi matrix.

Differentiating equations (1.19)–(1.20), we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi^{(\beta)}(x) = P^{(\beta)}\Phi^{(\beta)}(x), \qquad \frac{\mathrm{d}}{\mathrm{d}x}\Psi^{(\beta)}(x) = P^{(\beta)}\Psi^{(\beta)}(x), \qquad \beta = 1, 4, \tag{2.1}$$

$$x\frac{\mathrm{d}}{\mathrm{d}x}\Phi^{(4)}(x) = R^{(4)}\Phi^{(4)}(x), \qquad x\frac{\mathrm{d}}{\mathrm{d}x}\Psi^{(4)}(x) = [R^{(4)} - \mathbf{1}]\Psi^{(4)}(x), \tag{2.2}$$

$$x\frac{\mathrm{d}}{\mathrm{d}x}\Psi^{(1)}(x) = R^{(1)}\Psi^{(1)}(x), \qquad x\frac{\mathrm{d}}{\mathrm{d}x}\Phi^{(1)}(x) = [R^{(1)} - \mathbf{1}]\Phi^{(1)}(x), \tag{2.3}$$

where $\Phi^{(\beta)}(x)$ and $\Psi^{(\beta)}(x)$ are the semi-infinite column vectors defined in equations (1.4) and (1.5). This is equivalent to saying

$$[R^{(4)} - xP^{(4)}]\Phi^{(4)}(x) = 0, \qquad [R^{(4)} - xP^{(4)} - \mathbf{1}]\Psi^{(4)}(x) = 0, \tag{2.4}$$

$$[R^{(1)} - xP^{(1)}]\Psi^{(1)}(x) = 0, \qquad [R^{(1)} - xP^{(1)} - \mathbf{1}]\Phi^{(1)}(x) = 0. \tag{2.5}$$

These finite band matrices satisfy the following commutation relations:

$$[Q^{(\beta)}, P^{(\beta)}] = \mathbf{1}, \qquad [R^{(\beta)}, P^{(\beta)}] = P^{(\beta)}. \tag{2.6}$$

Here, each entry is a 2×2 quaternion of the form

$$A_{n,m} := \begin{bmatrix} \tilde{A}_{2n,2m} & \tilde{A}_{2n,2m+1} \\ \tilde{A}_{2n+1,2m} & \tilde{A}_{2n+1,2m+1} \end{bmatrix}. \tag{2.7}$$

Using $\left(\psi_n^{(4)}, \psi_m^{(4)}\right)$ and $\left(x\psi_n^{(4)}, \psi_m^{(4)}\right)$ for $\beta = 4$, and replacing $\psi^{(4)}(x)$ by $\phi^{(1)}(x)$ for $\beta = 1$, we get

$$P^{(\beta)} = -P^{(\beta)D}, \qquad R^{(\beta)} = -R^{(\beta)D}, \tag{2.8}$$

where dual of a matrix A is defined as [27]

$$A^D = -ZA^tZ. \tag{2.9}$$

Remark. The condition of anti-self-duality imposes a much lighter restriction on a diagonal quaternion than antisymmetry condition on a diagonal matrix element. For example, it leaves the off-diagonal entry of the diagonal quaternion arbitrary. This is the reason why the odd skew-orthogonal polynomials are arbitrary upto the addition of a lower even polynomial.

However, starting with $\left(x\phi_n^{(\beta)}, \psi_m^{(\beta)}\right)$ and using (2.6), we get

$$\left(Q^{(\beta)} - Q^{(\beta)D}\right)P^{(\beta)} = P^{(\beta)}\left(Q^{(\beta)} - Q^{(\beta)D}\right) = \mathbf{1}. \tag{2.10}$$

As pointed out in the beginning of the section, this essentially means that like the orthogonal polynomials we do not have a tri-diagonal Jacobi matrix for skew-orthogonal polynomials. It is this relation that causes the generalized Christoffel–Darboux sum to have a local behaviour (i.e., the rank has dependence on the measure or weight function). In a sense, this is a major setback in our hope of defining a 2×2 matrix Riemann–Hilbert problem for skew-orthogonal vectors, similar to that of the orthogonal polynomials. Also from the dependence of $P^{(\beta)}$ and $R^{(\beta)}$ on $2d$ (the degree of the polynomial $V(x)$), we can conclude that the 'size' of our q-matrix Riemann–Hilbert problem will depend on d.

From [5], we also have

$$P^{(\beta)} + [V'(Q^{(\beta)})] = \text{lower}, \qquad R^{(\beta)} + [Q^{(\beta)}V'(Q^{(\beta)})] = \text{lower}_+, \tag{2.11}$$

where 'lower' and 'lower$_+$' denote a strictly lower triangular matrix and a lower triangular matrix with the principal diagonal, respectively.

Having obtained the recursion relations for the skew-orthonormal vectors, we introduce a convenient notation to express band matrices $P^{(\beta)}$ and $R^{(\beta)}$:

$$\begin{aligned} \sum_{m=n-d}^{n+d} P_{nm}^{(\beta)}\Phi_m^{(\beta)}(x) &= \sum_{k=-d}^{d} \zeta_k^{(\beta)}(n)\Phi_{n-k}^{(\beta)}(x), \\ \sum_{m=n-d}^{n+d} R_{nm}^{(\beta)}\Phi_m^{(\beta)}(x) &= \sum_{k=-d}^{d} \eta_k^{(\beta)}(n)\Phi_{n-k}^{(\beta)}(x), \end{aligned} \tag{2.12}$$

and

$$\sum_{m=n-d}^{n+d} R_{nm}^{(\beta)}\Phi_m^{(\beta)}(x) - x\sum_{m=n-d}^{n+d} P_{nm}^{(\beta)}\Phi_m^{(\beta)}(x) = \sum_{k=-d}^{d}\alpha_k^{(\beta)}(n)\Phi_{n-k}^{(\beta)}(x), \tag{2.13}$$

where

$$\begin{aligned} \eta_k^{(\beta)} &:= \mathrm{diag}\big(\eta_k^{(\beta)}(0), \ldots, \eta_k^{(\beta)}(n), \ldots\big), \\ \zeta_k^{(\beta)} &:= \mathrm{diag}\big(\zeta_k^{(\beta)}(0), \ldots, \zeta_k^{(\beta)}(n), \ldots\big), \\ \alpha_k^{(\beta)} &:= \mathrm{diag}\big(\alpha_k^{(\beta)}(0), \ldots, \alpha_k^{(\beta)}(n), \ldots\big). \end{aligned} \tag{2.14}$$

Here, we have suppressed the x-dependence of $\alpha_k^{(\beta)}(n)$. This notation will be useful in deriving the folding function.

More explicitly, with this notation, we can write

$$P^{(\beta)} = \left[\begin{array}{cccccccc} \zeta_0^{(\beta)}(0) & \zeta_{-1}^{(\beta)}(0) & \cdots & \cdots & \zeta_{-d}^{(\beta)}(0) & 0 & 0 & \cdots \\ \zeta_1^{(\beta)}(1) & \zeta_0^{(\beta)}(1) & \zeta_{-1}^{(\beta)}(1) & \cdots & \zeta_{-d+1}^{(\beta)}(1) & \zeta_{-d}^{(\beta)}(1) & 0 & \cdots \\ \zeta_2^{(\beta)}(2) & \zeta_1^{(\beta)}(2) & \zeta_0^{(\beta)}(2) & \cdots & \zeta_{-d+2}^{(\beta)}(2) & \zeta_{-d+1}^{(\beta)}(2) & \zeta_{-d}^{(\beta)}(2) & 0 \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \ddots \\ \zeta_d^{(\beta)}(d) & \zeta_{d-1}^{(\beta)}(d) & \cdots & \cdots & \zeta_0^{(\beta)}(d) & \cdots & \cdots & \cdots \\ 0 & \zeta_d^{(\beta)}(d+1) & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\ 0 & 0 & \zeta_d^{(\beta)}(d+2) & \ddots & \ddots & \ddots & & \end{array}\right), \tag{2.15}$$

and

$$R^{(\beta)} = \left[\begin{array}{cccccccc} \eta_0^{(\beta)}(0) & \eta_{-1}^{(\beta)}(0) & \cdots & \cdots & \eta_{-d}^{(\beta)}(0) & 0 & 0 & \cdots \\ \eta_1^{(\beta)}(1) & \eta_0^{(\beta)}(1) & \eta_{-1}^{(\beta)}(1) & \cdots & \eta_{-d}^{(\beta)}(1) & \eta_{-d}^{(\beta)}(1) & 0 & \cdots \\ \eta_2^{(\beta)}(2) & \eta_1^{(\beta)}(2) & \eta_0^{(\beta)}(2) & \cdots & \eta_{-d+1}^{(\beta)}(2) & \eta_{-d}^{(\beta)}(2) & \eta_{-d}^{(\beta)}(2) & 0 \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \ddots \\ \eta_d^{(\beta)}(d) & \eta_{d-1}^{(\beta)}(d) & \cdots & \cdots & \eta_0^{(\beta)}(d) & \cdots & \cdots & \cdots \\ 0 & \eta_d^{(\beta)}(d+1) & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\ 0 & 0 & \eta_d^{(\beta)}(d+2) & \ddots & \ddots & \ddots & & \end{array}\right). \tag{2.16}$$

We note that for even $V(x)$, the quaternions of the outermost band are such that

$$\forall n, \qquad \zeta_{\pm d}^{(\beta)}(n) \equiv \begin{bmatrix} 0 & 0 \\ \big(\zeta_{\pm d}^{(\beta)}(n)\big)_{10} & 0 \end{bmatrix}, \qquad \eta_{\pm d}^{(\beta)}(n) \equiv \begin{bmatrix} \big(\eta_{\pm d}^{(\beta)}(n)\big)_{00} & 0 \\ \big(\eta_{\pm d}^{(\beta)}(n)\big)_{10} & \big(\eta_{\pm d}^{(\beta)}(n)\big)_{11} \end{bmatrix}. \tag{2.17}$$

From (2.6), we get the following quadratic relation between the coefficients $\big(\zeta_j^{(\beta)}, \eta_j^{(\beta)}\big)$:

$$\sum_{j=l}^{d-1}\eta_{l-j}^{(\beta)}(n)\zeta_j^{(\beta)}(n-l+j) - \sum_{j=l}^{d-1}\zeta_{l-j}^{(\beta)}(n)\eta_j^{(\beta)}(n-l+j) = \zeta_l^{(\beta)}(n). \tag{2.18}$$

This is the compatibility relation for the difference-differential system and will be used in section 9.

Remark. Here, one can note the basic difference between the properties of bi-orthogonal polynomials and their skew-orthogonal counterparts. In the skew-orthogonal vectors, the semi-infinite matrices are symmetric around the principal diagonal. This is not true for the bi-orthogonal polynomials where the matrices have an asymmetry.

3. The deformation matrix

In this section, we will consider infinitesimal deformation corresponding to changes in u_K, the coefficient of the potential $V(x)$. Using the definition of the semi-infinite matrices $\Phi^{(\beta)}(x)$ and $\widehat{\Phi}^{(\beta)}(x)$ (and $\Psi^{(\beta)}(x)$, $\widehat{\Psi}^{(\beta)}(x)$), the deformation matrix can be defined as

$$\frac{\partial}{\partial u_K}\Phi^{(\beta)}(x) = U_K^{(\beta)}\Phi^{(\beta)}(x), \qquad \frac{\partial}{\partial u_K}\widehat{\Phi}^{(\beta)}(x) = \widehat{\Phi}(x)\big[U_K^{(\beta)}\big]^D, \qquad K = 1,\ldots,2d, \tag{3.1}$$

$$\frac{\partial}{\partial u_K}\Psi^{(\beta)}(x) = U_K^{(\beta)}\Psi^{(\beta)}(x), \qquad \frac{\partial}{\partial u_K}\widehat{\Psi}^{(\beta)}(x) = \widehat{\Psi}(x)\big[U_K^{(\beta)}\big]^D, \qquad K = 1,\ldots,2d. \tag{3.2}$$

The matrix $U_K^{(\beta)}$ is anti-self-dual, i.e.,

$$U_K^{(\beta)} = -\big[U_K^{(\beta)}\big]^D. \tag{3.3}$$

Moreover, they satisfy the following relations with P and R (we drop the superscript β from the matrix $Q^{(\beta)}$, $P^{(\beta)}$ and $R^{(\beta)}$ for simplicity):

$$\partial u_K P = \big[U_K^{(\beta)}, P\big], \qquad \partial u_K R = \big[U_K^{(\beta)}, R\big]. \tag{3.4}$$

Explicitly, $U_K^{(\beta)}$ can be written as

$$U_K^{(\beta)} = -\frac{1}{K}\big((Q^K)_+ - (Q^K)_-^D\big) - \frac{1}{2K}\big((Q^K)_0 - (Q^K)^D{}_0\big), \tag{3.5}$$

where Q_+, Q_- denote the upper and lower triangular quaternion matrices while Q_0 is the diagonal quaternion.

Proof. Differentiating equation (1.12) w.r.t. u_K and using equations (3.1) and (3.2), we get equation (3.3). Equation (3.4) follows by interchanging the operators $\partial/\partial x$ and $x\partial/\partial x$ with $\partial/\partial u_K$. Finally to prove equation (3.5), we start with the (2×1) normalized quasi-polynomial

$$\Phi_n^{(\beta)}(x) := \frac{1}{\sqrt{g_n^{(\beta)}}}\mathrm{e}^{-V(x)}\Pi_n(x), \qquad n \in \mathbb{N}, \tag{3.6}$$

where $V(x)$ is defined in equation (1.2) and

$$\Pi_n(x) = \begin{pmatrix} \pi_{2n}(x) \\ \pi_{2n+1}(x) \end{pmatrix} \tag{3.7}$$

is the monic skew-orthogonal polynomial, defined in equation (1.9). To save cluttering, we have suppressed the β dependence of $\pi_n(x)$. We have used $g_{2n}^{(\beta)} = g_{2n+1}^{(\beta)}$. Differentiating with respect to the deformation parameter, we get

$$\partial u_K \Phi_n^{(\beta)}(x) = -\frac{1}{2g_n^{(\beta)}}\partial u_K\big(g_n^{(\beta)}\big)\Phi_n^{(\beta)}(x) + \frac{\mathrm{e}^{-V(x)}}{\sqrt{g_n^{(\beta)}}}\partial u_K \Pi_n(x) - \frac{x^K}{K}\Phi_n^{(\beta)}(x). \tag{3.8}$$

Also differentiating the skew-normalization condition, we get

$$-\frac{\partial}{\partial u_K} g_n^{(\beta)} = -\int \left[\partial u_K \left(\Phi_n^{(\beta)}(x)\right)\widehat{\Psi}_n^{(\beta)}(x) + \Phi_n^{(\beta)}(x)\partial u_K \widehat{\Psi}_n^{(\beta)}(x)\right] \mathrm{d}x,$$
$$= \frac{1}{K}[Q^K + (Q^K)^D]_{n,n} g_n^{(\beta)}. \tag{3.9}$$

The diagonal elements have the form

$$\left(U_K^{(\beta)}\right)_{n,n} = -\left(\frac{Q^K}{K}\right)_{n,n} + \frac{1}{2K}\left((Q^K)_{n,n} + (Q^K)_{n,n}^D\right),$$
$$= -\frac{1}{2K}\left[Q_{n,n}^K - (Q^K)_{n,n}^D\right]. \tag{3.10}$$

Thus, we get

$$U_K^{(\beta)} = -\frac{1}{K}\left(Q_+^K - (Q^K)_-^D\right) - \frac{1}{2K}\left(\left(Q_0^K\right) - \left(Q_0^K\right)^D\right). \tag{3.11}$$

□

Remark. Here, the operation $(Q^K)_-^D$ means that the dual is taken first and then the lower triangular part collected. Also, due to the arbitrariness in the definition of an anti-self-dual matrix, we may choose the lower off-diagonal element of the diagonal quaternions in $U_K^{(\beta)}$ zero. With this choice, we can remove the arbitrariness in the definition of the odd skew-orthonormal polynomials.

4. The difference equation

In this section, we introduce a sequence of companion-like matrices $A_N^{(\beta)}(x)$ and $B_N^{(\beta)}(x)$ of sizes $(2d) \times (2d)$. Using the relations in equations (2.4) and (2.5), we can write

$$\sum_{k=-d}^{d} \alpha_k^{(4)}(n)\Phi_{n-k}^{(4)}(x) = 0 \qquad \sum_{k=-d}^{d} \alpha_k^{(1)}(n)\Psi_{n-k}^{(1)}(x) = 0 \qquad \forall k \in \mathbb{N}, \tag{4.1}$$

and

$$\sum_{k=-d}^{d} \left(\alpha_k^{(1)}(n) - \delta_{0,k}\right)\Phi_{n-k}^{(1)}(x) = 0 \qquad \sum_{k=-d}^{d} \left(\alpha_k^{(4)}(n) - \delta_{0,k}\right)\Psi_{n-k}^{(4)}(x) = 0 \qquad \forall k \in \mathbb{N}, \tag{4.2}$$

we get for $N \geqslant d$

$$\Phi_{W+1}^{(4)}(x) := \begin{pmatrix} \Phi_{N-d+1}^{(4)}(x) \\ \vdots \\ \vdots \\ \Phi_{N+d}^{(4)}(x) \end{pmatrix} = A_N^{(4)}(x) \begin{pmatrix} \Phi_{N-d}^{(4)}(x) \\ \vdots \\ \vdots \\ \Phi_{N+d-1}^{(4)}(x) \end{pmatrix} = A_N^{(4)}(x)\Phi_W^{(4)}(x), \tag{4.3}$$

and

$$\Psi_{W+1}^{(1)}(x) := \begin{pmatrix} \Psi_{N-d+1}^{(1)}(x) \\ \vdots \\ \vdots \\ \Psi_{N+d}^{(1)}(x) \end{pmatrix} = A_N^{(1)}(x) \begin{pmatrix} \Psi_{N-d}^{(1)}(x) \\ \vdots \\ \vdots \\ \Psi_{N+d-1}^{(1)}(x) \end{pmatrix} = A_N^{(1)}(x)\Psi_W^{(1)}(x), \tag{4.4}$$

where (suppressing the β dependence of $\alpha_j(n)$)

$$A_N^{(\beta)}(x) = \begin{pmatrix} 0 & \mathbf{1} & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{1} & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \mathbf{1} \\ -(\alpha_{-d}(N))^{-1}\alpha_d(N) & \dots & -(\alpha_{-d}(N))^{-1}(\alpha_0(N)) & \dots & -(\alpha_{-d}(N))^{-1}\alpha_{-d+1}(N) \end{pmatrix}. \tag{4.5}$$

Remark. For potential $V(x)$ with even degree and $\forall n$, $\alpha_{\pm d}(n) = R^{(\beta)}_{n,n\pm d} - xP^{(\beta)}_{n,n\pm d}$ and its inverse has the quaternion structure:

$$\alpha_{\pm d}(n) := \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}; \qquad \alpha_{\pm d}(n)^{-1} = \begin{pmatrix} \frac{1}{a} & 0 \\ -\frac{b}{ac} & \frac{1}{c} \end{pmatrix}, \qquad a, c \neq 0. \tag{4.6}$$

Hence, they are invertible. This is the criterion for the existence of the skew-orthogonal polynomials corresponding to even potential. We leave it to the reader to verify that $\alpha_{\pm d}(n)$ is non-invertible for potentials with odd degree.

Using this, one can easily see that

$$\Phi^{(4)}_{W-1}(x) := \begin{pmatrix} \Phi^{(4)}_{N-d-1}(x) \\ \vdots \\ \vdots \\ \Phi^{(4)}_{N+d-2}(x) \end{pmatrix} = \left(A^{(4)}_{N-1}(x)\right)^{-1} \begin{pmatrix} \Phi^{(4)}_{N-d}(x) \\ \vdots \\ \vdots \\ \Phi^{(4)}_{N+d-1}(x) \end{pmatrix} = \left(A^{(4)}_{N-1}(x)\right)^{-1} \Phi^{(4)}_W(x), \tag{4.7}$$

and

$$\Psi^{(1)}_{W-1}(x) := \begin{pmatrix} \Psi^{(1)}_{N-d-1}(x) \\ \vdots \\ \vdots \\ \Psi^{(1)}_{N+d-2}(x) \end{pmatrix} = \left(A^{(1)}_{N-1}(x)\right)^{-1} \begin{pmatrix} \Psi^{(1)}_{N-d}(x) \\ \vdots \\ \vdots \\ \Psi^{(1)}_{N+d-1}(x) \end{pmatrix} = \left(A^{(1)}_{N-1}(x)\right)^{-1} \Psi^{(1)}_W(x), \tag{4.8}$$

where

$$\left(A^{(\beta)}_{N-1}(x)\right)^{-1} = \begin{pmatrix} -(\alpha_d(N-1))^{-1}\alpha_{d-1}(N-1) & \dots & -(\alpha_d(N-1))^{-1}(\alpha_0(N-1)) & \dots & -(\alpha_d(N-1))^{-1}\alpha_{-d}(N-1) \\ \mathbf{1} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \mathbf{1} & 0 \end{pmatrix}. \tag{4.9}$$

Similarly, for $\Psi^{(4)}(x)$ and $\Phi^{(1)}(x)$, $A^{(\beta)}(x)$ is replaced by $B^{(\beta)}(x)$, where

$$B_N^{(\beta)}(x) = \begin{pmatrix} 0 & \mathbf{1} & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{1} & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \mathbf{1} \\ -(\alpha_{-d}(N))^{-1}\alpha_d(N) & \dots & -(\alpha_{-d}(N))^{-1}(\alpha_0(N)-1) & \dots & -(\alpha_{-d}(N))^{-1}\alpha_{-d+1}(N) \end{pmatrix}. \tag{4.10}$$

More generally, this ladder operator can be used successively to obtain

$$\Phi^{(\beta)}_{W-j}(x) = \left(A^{(\beta)}_{N-j}(x)\right)^{-1} \cdots \left(A^{(\beta)}_{N-1}(x)\right)^{-1} \Phi^{(\beta)}_{W}(x), \qquad \forall j \in \mathbb{N}, \tag{4.11}$$

and

$$\Phi^{(\beta)}_{W+j}(x) = A^{(\beta)}_{N+j-1}(x) \cdots A^{(\beta)}_{N}(x)\Phi^{(\beta)}_{W}(x), \qquad \forall j \in \mathbb{N}. \tag{4.12}$$

This is the underlying idea behind folding which will be discussed in details in the next section.

5. Folding

Remark. From this point on we focus on the skew-orthogonal vectors $\Phi^{(4)}_n(x)$ (and its dual $\Psi^{(1)}_n(x)$), but everything being said can be immediately extended to the vectors $\Phi^{(1)}_n(x)$ (and hence $\Psi^{(4)}_n(x)$) by interchanging the rôles of matrices $R^{(\beta)}$ by $R^{(\beta)} - \mathbf{1}$.

The notion of 'folding' is the following: we express any quasi-polynomial $\Phi^{(\beta)}_n(x)$ (or $\Psi^{(\beta)}_n(x)$) as a linear combination of $2d$ *fixed* consecutive vectors $\Phi^{(\beta)}_W(x) \equiv \Phi^{(4)}_{N-j}(x)$, $j = d, \ldots, -d+1$, *with polynomial coefficients*. We now provide a way of computing the folding function on the same line as done in [20] for bi-orthogonal polynomials.

For a fixed subsequence (or window) W, such that $W \Rrightarrow \{N-d, \ldots, N+d-1\}$, $N \geqslant d$, we seek to describe the folding of the infinite wave vector $\Phi^{(\beta)}(x)$ onto the window W by means of a single quaternion matrix $F(x)$ of size $\infty \times (2d)$ with polynomial entries such that

$$\forall n, \qquad \Phi^{(\beta)}_n(x) = \sum_{k=N-d}^{N+d-1} F_{n,k}(x)\Phi^{(\beta)}_k(x). \tag{5.1}$$

(Similarly for $\Psi^{(\beta)}_n(x)$.) In fact, it is more convenient to think of $F(x)$ as a $\infty \times \infty$ matrix with only a vertical band of width $2d$ of nonzero entries (with column index in the range $N-d, \ldots, N+d-1$).

Now, we will find an explicit form of the 'folding function' in terms of some upper and lower triangular matrices. We will derive the folding function for $\Phi^{(4)}(x)$ and $\Psi^{(1)}(x)$. For the folding function for $\Psi^{(4)}(x)$ and $\Phi^{(1)}(x)$ we follow the same procedure and hence state the result without proof. Note that everything will be written in terms of quaternion matrices.

We will start with a few definitions and identities. We will introduce the shift matrix Λ comprising of 2×2 blocks of unit matrix:

$$\Lambda = \begin{pmatrix} 0 & \mathbf{1} & 0 & \cdots \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.2}$$

satisfying the relation

$$\Lambda^t = \Lambda^D, \tag{5.3}$$

i.e., in this case, the transpose is the same as dual. Hence for convenience, for the rest of the paper, we will denote it as Λ^t.

Let us define the projection matrices:

$$\prod^{N} := \mathbf{1} - \prod_{N}, \qquad \prod_{M}^{N} := \prod_{N} - \prod_{M}. \tag{5.4}$$

We will also use the following identities:

$$\prod^{N+d}(\Lambda^t)^d = (\Lambda^t)^d \prod^{N}, \qquad \prod_{N+d}(\Lambda^t)^d = (\Lambda^t)^d \prod_{N}, \qquad \prod^{N+d}\prod_{N+d} = 0, \qquad \prod^{N+d}\prod_{N} = 0, \tag{5.5}$$

$$\prod^{N}\prod_{N+d} = \prod_{N+d} - \prod_{N} = \prod_{N}^{N+d}, \qquad (\Lambda^t)^d\Lambda^d = 1 - \prod_{d}, \qquad \Lambda^d \prod_{d} = 0. \tag{5.6}$$

To find an explicit formula for the matrix $F(x)$, we will use the diagonal band matrices α_j, $j = -d, \dots, d$, to express $R^{(\beta)} - xP^{(\beta)}$:

$$\overline{R}^{(\beta)}(x) = R^{(\beta)} - xP^{(\beta)} = \alpha_{-d}\Lambda^d + \sum_{k=1}^{d-1}[\alpha_{-k}\Lambda^k] + \sum_{k=0}^{d}[\alpha_k(\Lambda^t)^k], \tag{5.7}$$

where Λ is the shift matrix defined in equation (5.2). (For simplicity, we have again suppressed the β dependence of α_j.)

Using the shift matrix in equation (5.7), we get

$$(\Lambda^t)^d\alpha_{-d}^{-1}[\overline{R}^{(\beta)}(x)] = (\Lambda^t)^d(\Lambda)^d + (\Lambda^t)^d\alpha_{-d}^{-1}\sum_{k=1}^{d-2}[\alpha_{-k}\Lambda^k] + (\Lambda^t)^d\alpha_{-d}^{-1}\sum_{k=0}^{d}[\alpha_k(\Lambda^t)^k]. \tag{5.8}$$

Thus, we can define a semi-infinite matrix $G^{(\beta)}$, which is strictly lower triangular:

$$G^{(\beta)} := 1 - \prod_{d} - (\Lambda^t)^d\alpha_{-d}^{-1}[\overline{R}^{(\beta)}(x)]. \tag{5.9}$$

Similarly,

$$\Lambda^d\alpha_d^{-1}[\overline{R}^{(\beta)}(x)] = 1 + \Lambda^d\alpha_d^{-1}\left[\sum_{k=1}^{d}\alpha_{-k}\Lambda^k + \sum_{k=0}^{d-1}\alpha_k(\Lambda^t)^k\right] \tag{5.10}$$

allows us to define a strictly upper triangular matrix:

$$C^{(\beta)} := 1 - \Lambda^d\alpha_d^{-1}\overline{R}^{(\beta)}(x). \tag{5.11}$$

Multiplying equation (5.9) by $\alpha_{-d}(\Lambda)^d$ on the left-hand side and $(1 - G^{(\beta)})^{-1}(\Lambda^t)^d\alpha_{-d}^{-1}$ on the right-hand side, we get

$$\mathbf{1} = \overline{R}^{(\beta)}(x)\left[(1 - G^{(\beta)})^{-1}(\Lambda^t)^d\alpha_{-d}^{-1}\right], \tag{5.12}$$

while for the upper triangular matrix we have the relation

$$\mathbf{1} = (1 - C^{(\beta)})^{-1}\Lambda^d\alpha_d^{-1}\overline{R}^{(\beta)}(x). \tag{5.13}$$

Thus, the matrix $[R^{(\beta)} - xP^{(\beta)}]$ has a left and a right inverse, but they are not the same. One is upper triangular while the other is lower triangular. However, they satisfy the following relation:

$$[\overline{R}^{(\beta)}(x)]_L^{-1} = -\left([\overline{R}^{(\beta)}(x)]_R^{-1}\right)^D. \tag{5.14}$$

One observes similar features for the matrix $(Q - x)$ in the context of the two-matrix model studied in [20].

Following the above procedure, we can also get the upper and lower triangular matrices, needed to fold $\Psi^{(4)}(x)$ and $\Phi^{(1)}(x)$. They are

$$\tilde{G}^{(\beta)} := 1 - \prod_{d} -(\Lambda^t)^d \alpha_{-d}^{-1}[\overline{R}^{(\beta)}(x) - \mathbf{1}], \qquad \tilde{C}^{(\beta)} := 1 - \Lambda^d \alpha_d^{-1}[\overline{R}^{(\beta)}(x) - \mathbf{1}], \tag{5.15}$$

such that they satisfy

$$\mathbf{1} = [[\overline{R}^{(\beta)}(x) - \mathbf{1}]]\big[(1 - \tilde{G}^{(\beta)})^{-1}(\Lambda^t)^d \alpha_{-d}^{-1}\big], \qquad \mathbf{1} = (1 - \tilde{C}^{(\beta)})^{-1}\Lambda^d \alpha_d^{-1}[\overline{R}^{(\beta)}(x) - \mathbf{1}]. \tag{5.16}$$

5.1. Upper folding

From the definition $[R^{(4)} - xP^{(4)}]\Phi^{(4)}(x) = 0$, and equations (5.9) and (5.11), we get

$$\Phi^{(4)}(x) = C^{(4)}\Phi^{(4)}(x), \qquad \Phi^{(4)}(x) = \left(G^{(4)} + \prod_{d+1}\right)\Phi^{(4)}(x). \tag{5.17}$$

Similarly, from the definition $[R^{(1)} - xP^{(1)}]\Psi^{(1)}(x) = 0$, and equations (5.9) and (5.11), we get

$$\Psi^{(1)}(x) = C^{(1)}\Psi^{(1)}(x), \qquad \Psi^{(1)}(x) = \left(G^{(1)} + \prod_{d+1}\right)\Psi^{(1)}(x). \tag{5.18}$$

Given that the quaternion matrix G is strictly lower triangular, we can use equation (5.18) in an iterative manner to get

$$\begin{aligned} \prod^{N}\Phi^{(4)}(x) &= (1 - G^{(4)})^{-1}\prod^{N} G^{(4)}\prod_{N}\Phi^{(4)}(x), \\ \prod^{N}\Psi^{(1)}(x) &= (1 - G^{(1)})^{-1}\prod^{N} G^{(1)}\prod_{N}\Psi^{(1)}(x). \end{aligned} \tag{5.19}$$

We will drop the parameter β for simplification. Also since the matrix G has d bands below the principal diagonal, we have for the folding function

$$\prod^{\overline{N}} F(x) = (1 - G)^{-1}\prod^{\overline{N}} G \prod_{\overline{N}}, \tag{5.20}$$

where $\overline{N} = N + d - 1$. Simplifying further, we replace G from equation (5.9) to get

$$\prod^{\overline{N}} F(x) = (1 - G)^{-1}\prod^{\overline{N}}\left[1 - \prod_{d} -\alpha_{-d}^{-1}(\Lambda^t)^d(R^{(\beta)} - xP^{(\beta)})\right]\prod_{\overline{N}}. \tag{5.21}$$

Then using the identities obtained before, one gets

$$\begin{aligned} \prod^{\overline{N}} F(x) &= -(1 - G)^{-1}\prod^{\overline{N}}\alpha_{-d}^{-1}(\Lambda^t)^d[\overline{R}^{(\beta)}(x)]\prod_{\overline{N}}, \\ &= -\left[(1 - G)^{-1}\alpha_{-d}^{-1}(\Lambda^t)^d\prod^{N}(\overline{R}^{(\beta)}(x))\prod_{\overline{N}}\right], \\ &= -\prod_{N}^{\overline{N}} -(1 - G)^{-1}\alpha_{-d}^{-1}(\Lambda^t)^d\left[\overline{R}^{(\beta)}(x), \prod_{N}\right]. \end{aligned}$$

Thus, we have for the upper folding

$$\prod^{\overline{N}} F(x) = -\prod_{N}^{\overline{N}} -(1-G)^{-1}\alpha_{-d}^{-1}(\Lambda^t)^d \left[\overline{R}^{(\beta)}(x), \prod_N\right]. \tag{5.22}$$

5.2. Lower folding

Now, we will study the lower folding function. We can write

$$\begin{aligned} \prod_M \Phi^{(4)}(x) &= (1-C^{(4)})^{-1}\prod_M C^{(4)}\prod^M \Phi^{(4)}(x), \\ \prod_M \Psi^{(1)}(x) &= (1-C^{(1)})^{-1}\prod_M C^{(1)}\prod^M \Psi^{(1)}(x), \end{aligned} \tag{5.23}$$

where $M = N-d$. This relation is valid since we are folding finite band matrices. Thus, $F(x)$ can be written as

$$\prod_M F(x) = (1-C)^{-1}\prod_M C\prod^M. \tag{5.24}$$

Replacing C from equation (5.11) and using the identities, we get

$$\begin{aligned} \prod_M F(x) &= -(1-C)^{-1}\Lambda^d\alpha_d^{-1}\prod_N \overline{R}^{(\beta)}(x)\prod^M, \\ &= -\prod_M^N +(1-C)^{-1}\Lambda^d\alpha_d^{-1}\left[\overline{R}^{(\beta)}(x), \prod_N\right], \\ &= -\prod_M^N +(1-C)^{-1}\Lambda^d\alpha_d^{-1}\left[\overline{R}^{(\beta)}(x), \prod_N\right]. \end{aligned} \tag{5.25}$$

5.3. Folding function

The folding function is

$$F(x) = \prod_M F(x) + \prod^{\overline{N}} F(x) + \prod_M^N, \tag{5.26}$$

such that we get

$$\boxed{F(x) = \left((1-C)^{-1}\Lambda^d\alpha_d^{-1} - (1-G)^{-1}\alpha_{-d}^{-1}(\Lambda^t)^d\right)\left[\overline{R}^{(\beta)}(x), \textstyle\prod_N\right].} \tag{5.27}$$

One can easily repeat the above calculations to find the folding function for $\Psi^{(4)}(x)$ and $\Phi^{(1)}(x)$. It gives

$$\boxed{\tilde{F}(x) = \left((1-\tilde{C})^{-1}\Lambda^d\alpha_d^{-1} - (1-\tilde{G})^{-1}\alpha_{-d}^{-1}(\Lambda^t)^d\right)\left[\overline{R}^{(\beta)}(x), \textstyle\prod_N\right].} \tag{5.28}$$

Remark. In the context of bi-orthogonal polynomials, Bergere and Eynard [37] have found a very elegant expression for the folding function. It would be interesting to see the existence of such compact expression in the context of skew-orthogonal vectors.

6. Folded deformation matrices

In this section, we will show that a finite subsequence or window of these skew-orthonormal vectors satisfies a system of PDEs under the infinitesimal change of the deformation parameter. We will prove for $\Phi^{(4)}(x)$ and $\Psi^{(1)}(x)$. It can be trivially extended to $\Psi^{(4)}(x)$ and $\Phi^{(1)}(x)$. We state the result.

For convenience, in this section, we will drop the superscript β from $Q^{(\beta)}$ as it has no importance on the deformation system. We define the folded deformation matrix:

$$\prod_M^{\overline{N}} U_K^{(\beta)} F(x) := U_K^{N(\beta)}, \qquad \prod_M^{\overline{N}} U_K^{(\beta)} \tilde{F}(x) := \underline{U}_K^{N(\beta)}, \qquad K = 1, \ldots, 2d, \tag{6.1}$$

where we project a $2d \times \infty$ quaternion matrix $\prod_M^{\overline{N}} U_K^{(\beta)}$ onto $\infty \times 2d$ quaternion matrices $F(x)$ and $\tilde{F}(x)$. We have

$$\begin{aligned}
-KU_K^{N(\beta)} &= \prod_M^{\overline{N}} \left[\left[Q_+^K - (Q^K)_-^D \right] + \frac{1}{2}\left[Q_0^K - \left(Q_0^K\right)^D \right] \right] F(x), \\
&= \prod_M^{\overline{N}} \left[\left[Q_+^K - (Q^K)_-^D \right] + \frac{1}{2}\left[Q_0^K - \left(Q_0^K\right)^D \right] \right] \left[\prod_M F(x) + \prod^{\overline{N}} F(x) + \prod_M^{\overline{N}} \right], \\
&= -\prod_M^{\overline{N}} \left[KU_K^{(\beta)} \right] \prod_M^{\overline{N}} + \prod_M^{\overline{N}} Q^K \prod^{\overline{N}} F(x) - \prod_M^{\overline{N}} (Q^K)^D \prod_M F(x), \\
&= -\prod_M^{\overline{N}} \left[KU_K^{(\beta)} \right] \prod_M^{\overline{N}} - \prod_M^{\overline{N}} Q^K \left[\prod_N^{\overline{N}} + (1-G)^{-1} \left(\Lambda^t \alpha_{-d}^{-1} \right)^d \left[\overline{R}^{(\beta)}(x), \prod_N \right] \right] \\
&\quad - \prod_M^{\overline{N}} (Q^K)^D \left[-\prod_M^{N} + (1-C)^{-1} \left(\alpha_{-d}^{-1} \Lambda \right)^d \left[\overline{R}^{(\beta)}(x), \prod_N \right] \right], \\
&= -\prod_M^{\overline{N}} \left[KU_K^{(\beta)} \right] \prod_M^{\overline{N}} - \prod_M^{\overline{N}} Q^K \prod_N^{\overline{N}} + \prod_M^{\overline{N}} (Q^K)^D \prod_M^{N} \\
&\quad - \prod_M^{\overline{N}} \left[(Q^K)(1-G)^{-1} \left(\Lambda^t \alpha_{-d}^{-1} \right)^d \left[\overline{R}^{(\beta)}(x), \prod_N \right] \right] \\
&\quad - \prod_M^{\overline{N}} \left[\left(Q^{K\,D} \right)(1-C)^{-1} \left(\alpha_{-d}^{-1} \Lambda \right)^d \left[P^{(\beta)}, \prod_N \right] \right], \\
&= \prod_M^{N} \frac{\left(Q_0^K + (Q_0^K)^D \right)}{2} \prod_M^{N} - \prod_{N+1}^{\overline{N}} \frac{\left(Q_0^K + \left(Q_0^K\right)^D \right)}{2} \prod_{N+1}^{\overline{N}} - 2x^K \left[\prod_M^{N} - \prod_N^{\overline{N}} \right] \\
&\quad - \prod_{N+1}^{\overline{N}} [Q^K - x^K] (\overline{R}^{(\beta)}(x))_R^{-1} \left[\left[\overline{R}^{(\beta)}(x), \prod_N \right] \right] \\
&\quad - \prod_M^{\overline{N}} (Q^K - x^K)^D (\overline{R}^{(\beta)}(x))_L^{-1} \left[\overline{R}^{(\beta)}(x), \prod_N \right],
\end{aligned}$$

$$= \prod_M^{\overline{N}} \frac{(Q_0^K + (Q_0^K)^D)}{2} \left[\prod_M^N - \prod_N^{\overline{N}} \right]$$
$$- \prod_M^{\overline{N}} \mathcal{W}_K(x) \left[\left[\overline{R}^{(\beta)}(x), \prod_N \right] \right] - 2x^K \left[\prod_M^N - \prod_N^{\overline{N}} \right], \tag{6.2}$$

where

$$\mathcal{W}_K(x) = (Q^K - x^K)(\overline{R}^{(\beta)}(x))_R^{-1} + (Q^K - x^K)^D (\overline{R}^{(\beta)}(x))_L^{-1}. \tag{6.3}$$

Now, if we define in analogy to the Pauli matrix σ_3,

$$\Sigma_3 := \left[\prod_M^N - \prod_N^{\overline{N}} \right] = \mathrm{diag}(\underbrace{\mathbf{1}\cdots\mathbf{1}}_{d}, \underbrace{-\mathbf{1}\cdots-\mathbf{1}}_{d}), \tag{6.4}$$

then we may write

$$\boxed{-K U_K^{N(\beta)} = \prod_M^{\overline{N}} \frac{\left(Q_0^K + \left(Q_0^K\right)^D\right)}{2} \Sigma_3 - \prod_M^{\overline{N}} \mathcal{W}_K(x) \left[\overline{R}^{(\beta)}(x), \prod_N \right] - 2x^K \Sigma_3.} \tag{6.5}$$

For $\Psi^{(4)}(x)$ and $\Phi^{(1)}(x)$, we have

$$\boxed{-K \underline{U}_K^{N(\beta)} = \prod_M^{\overline{N}} \frac{\left(Q_0^K + \left(Q_0^K\right)^D\right)}{2} \Sigma_3 - \prod_M^{\overline{N}} \underline{\mathcal{W}}_K(x) \left[\overline{R}^{(\beta)}(x), \prod_N \right] - 2x^K \Sigma_3,} \tag{6.6}$$

where

$$\underline{\mathcal{W}}_K(x) = (Q^K - x^K)(\overline{R}^{(\beta)}(x) - \mathbf{1})_R^{-1} + (Q^K - x^K)^D (\overline{R}^{(\beta)}(x) - \mathbf{1})_L^{-1}. \tag{6.7}$$

Thus, we see that these skew-orthogonal vectors satisfy a system of PDEs with respect to the deformation parameter and is given by

$$\frac{\partial}{\partial u_K} \Phi_W^{(4)}(x) = U_K^{N(4)} \Phi_W^{(4)}(x), \qquad \frac{\partial}{\partial u_K} \Psi_W^{(1)}(x) = U_K^{N(1)} \Psi_W^{(1)}(x), \tag{6.8}$$

$$\frac{\partial}{\partial u_K} \Psi_W^{(4)}(x) = \underline{U}_K^{N(4)} \Psi_W^{(4)}(x), \qquad \frac{\partial}{\partial u_K} \Phi_W^{(1)}(x) = \underline{U}_K^{N(1)} \Phi_W^{(1)}(x), \tag{6.9}$$

where $\Phi_W^{(\beta)}(x)$ and $\Psi_W^{(\beta)}(x)$ are defined in equations (1.26), (1.27).

7. The differential equation

Finally, using the results of the deformation equation, we will derive the ODEs satisfied by the finite subsequence of skew-orthogonal vectors.

Remark. We will explicitly calculate the differential operator for $\Phi^{(4)}(x)$ and $\Psi^{(4)}(x)$. The corresponding dual vectors $\Psi^{(1)}(x)$ and $\Phi^{(1)}(x)$, respectively, can be obtained by a simple transfer of $Q^{(4)} \mapsto Q^{(1)}$ and hence $P^{(4)} \mapsto P^{(1)}$ and $R^{(4)} \mapsto R^{(1)}$. Everything else remains the same.

We recall that the differential operator on any of these vectors is given as

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi^{(\beta)}(x) = P^{(\beta)}\Phi^{(\beta)}(x) = -\Big[(V'(Q))_+ - (V'(Q))_-^D + \frac{1}{2}\big[(V'(Q))_0 - (V'(Q))_0^D\big]\Big]\Phi^{(\beta)}(x)$$
$$= -\sum_{K=0}^{2d-1} u_{K+1}\left[Q_+^K - (Q^K)_-^D + \frac{1}{2}\big[Q_0^K - (Q^K)_0^D\big]\right]\Phi^{(\beta)}(x), \tag{7.1}$$

and similarly for $\Psi^{(\beta)}(x)$. Since $D_N^{(\beta)}$ and $\underline{D}_N^{(\beta)}$ are the folded version of $P^{(\beta)}$, we observe that the differential equation satisfied by the finite subsequence of $\Phi^{(4)}(x)$, namely $\Phi_W^{(4)}(x)$, is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi_W^{(4)}(x) = \prod_M^{\overline{N}} P^{(4)}F(x)\Phi^{(4)}(x),$$
$$= D_N^{(4)}\Phi_W^{(4)}(x), \tag{7.2}$$

such that with $Q := Q^{(4)}$, $P := P^{(4)}$ and $R := R^{(4)}$ we have

$$D_N^{(4)} = \prod_M^{\overline{N}} PF(x) = -\prod_M^{\overline{N}}\left[(V'(Q))_+ - (V'(Q))_-^D + \frac{1}{2}\big[(V'(Q))_0 - (V'(Q))_0^D\big]\right]F(x),$$
$$= -\sum_{K=0}^{2d-1} u_{K+1}\prod_M^{\overline{N}}\left[Q_+^K - (Q^K)_-^D + \frac{1}{2}\big[Q_0^K - (Q^K)_0^D\big]\right]F(x),$$
$$= -\sum_{K=0}^{2d-1} u_{K+1}\big(-KU_K^{N\,(4)}\big),$$
$$= 2V'(x)\Sigma_3 + \prod_M^{\overline{N}}\mathcal{W}_K^N(x)\left[\overline{R}(x), \prod_N\right]. \tag{7.3}$$

In the last step, we have used the anti-self-dual property of P.

Thus, we have

$$\boxed{\tfrac{\mathrm{d}}{\mathrm{d}x}\Phi_W^{(4)}(x) = \textstyle\prod_M^{\overline{N}} PF(x)\Phi^{(4)}(x) = \big[2V'(x)\Sigma_3 + \prod_M^{\overline{N}}\mathcal{W}_K^N(x)\big[\,\overline{R}(x), \prod_N\,\big]\big]\Phi_W^{(4)}(x).} \tag{7.4}$$

where

$$\mathcal{W}_{\mathcal{K}}^{\mathcal{N}}(x) = [V'(Q) - V'(x)](xP - R)_R^{-1} + [(V'(Q)) - V'(x)]^D(xP - R)_L^{-1}. \tag{7.5}$$

Replacing $\Phi^{(4)}(x) \mapsto \Psi^{(1)}(x)$ and $Q^{(4)} \mapsto Q^{(1)}$ (and hence $P^{(4)} \mapsto P^{(1)}$ and $R^{(4)} \mapsto R^{(1)}$), we can obtain the differential equation for the dual vector $\Psi^{(1)}(x)$.

Explicitly, we have

$$D_N^{(4)} = 2V'(x)\begin{pmatrix} \mathbf{1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \cdots & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{1} \end{pmatrix}_{2d\times 2d}$$

$$+\mathcal{W}_{\mathcal{K}}^{\mathcal{N}}(x)\begin{pmatrix} 0 & 0 & 0 & \alpha_{-d}(M) & 0 & 0 \\ 0 & 0 & 0 & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \alpha_{-1}(N) & \cdots & \alpha_{-d}(N) \\ -\alpha_d(N+1) & \cdots & -\alpha_1(N+1) & 0 & 0 & 0 \\ 0 & \ddots & \vdots & 0 & 0 & 0 \\ 0 & 0 & -\alpha_d(\overline{N}) & 0 & 0 & 0 \end{pmatrix}. \tag{7.6}$$

For $\Psi_W^{(4)}(x)$, we have

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\Psi_W^{(4)}(x) = \prod_M^{\overline{N}} P\tilde{F}(x)\Psi^{(4)}(x) = \left[2V'(x)\Sigma_3 + \prod_M^{\overline{N}} \underline{\mathcal{W}_K^N}(x)\left[\overline{R}(x), \prod_N\right]\right]\Psi_W^{(4)}(x),} \tag{7.7}$$

where $\underline{\mathcal{W}_{\mathcal{K}}^{\mathcal{N}}}(x)$ is given by

$$\underline{\mathcal{W}_{\mathcal{K}}^{\mathcal{N}}}(x) = [V'(Q) - V'(x)](\overline{R}(x) - 1)_R^{-1} + [(V'(Q)) - V'(x)]^D(\overline{R}(x) - 1)_L^{-1}. \tag{7.8}$$

For the dual vector $\Phi_W^{(1)}(x)$, we follow the same procedure. Replacing $\Psi^{(4)}(x) \mapsto \Phi^{(1)}(x)$ and $Q^{(4)} \mapsto Q^{(1)}$ (and hence $P^{(4)} \mapsto P^{(1)}$ and $R^{(4)} \mapsto R^{(1)}$), we can obtain the ODEs for the dual vector $\Phi^{(1)}(x)$.

8. Fundamental solutions

We have seen that a finite subsequence of (2×1) skew-orthogonal vectors $\Phi_{N-k}^{(\beta)}(x)$ and $\Psi_{N-k}^{(\beta)}(x), k = -d+1, \ldots, d$, satisfies a system of differential-difference-deformation equation. To obtain the fundamental solutions, useful in the Riemann–Hilbert analysis, we need to look for $2d$ other solutions. In this section, we will show that the Cauchy-like transforms of these skew-orthogonal vectors indeed form such solutions.

We will look for some integral representations of these skew-orthogonal vectors as the possible 'other solutions'. From the previous experience in orthogonal [10] and bi-orthogonal polynomials [21], we look for some Cauchy-like transforms $\tilde{\Psi}_n^{(\beta)}(x)$ and $\tilde{\Phi}_n^{(\beta)}(x)$ and hope that at least for sufficiently large n these functions will form the simultaneous solutions for the system of difference-differential-deformation equations. The remaining solutions can be obtained by taking the Cauchy-like transforms of certain moment functions $f_j(x)$ of order j, to be defined below. This trick had been used previously to define Riemann–Hilbert problems for bi-orthogonal [38] and skew-orthogonal polynomials [39], respectively.

Let us define

$$f_j(x) := \exp[V(x)]\int_{\mathbb{R}} \epsilon(x-y)y^j \exp[-V(y)]\,\mathrm{d}y, \qquad j = 0, \ldots, 2d-2. \tag{8.1}$$

We will show that the functions,

$$\Psi_n^{(j,\beta)}(x) := \int_{\mathbb{R}} \frac{f_j(x) - f_j(z)}{x-z}\Psi_n^{(\beta)}(z)\exp[-V(z)]\,\mathrm{d}z, \qquad j = 0, \ldots, 2d-2, \tag{8.2}$$

$$\Phi_n^{(j,\beta)}(x) := \int_{\mathbb{R}} \frac{f_j(x) - f_j(z)}{x-z}\Phi_n^{(\beta)}(z)\exp[-V(z)]\,\mathrm{d}z \qquad j = 0, \ldots, 2d-2, \tag{8.3}$$

and the Cauchy-like transform of the skew-orthogonal vectors,

$$\begin{aligned}\tilde{\Psi}_n^{(\beta)}(x) &= \exp[V(x)] \int_{\mathbb{R}} \frac{\Psi_n^{(\beta)}(z)\exp[-V(z)]}{(x-z)}\,\mathrm{d}z,\\ \tilde{\Phi}_n^{(\beta)}(x) &= \exp[V(x)] \int_{\mathbb{R}} \frac{\Phi_n^{(\beta)}(z)\exp[-V(z)]}{(x-z)}\,\mathrm{d}z, \qquad n\in\mathbb{N},\end{aligned} \tag{8.4}$$

are simultaneous solutions of the differential-difference-deformation equations.

8.1. The Cauchy-like transform

Componentwise, we get using equations (1.19) and (1.20)

$$\tilde{\psi}_n^{(4)}(x) = \sum P_{n,m}^{(4)}\tilde{\phi}_m^{(4)}(x), \qquad \tilde{\phi}_n^{(1)}(x) = \sum P_{n,m}^{(1)}\tilde{\psi}_m^{(1)}(x). \tag{8.5}$$

We also get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\psi}_n^{(\beta)}(x) &= \exp[V(x)]\int_{\mathbb{R}} \frac{V'(x)-V'(z)}{x-z}\psi_n^{(\beta)}(z)\exp[-V(z)]\,\mathrm{d}z + \sum P_{n,m}^{(\beta)}\tilde{\psi}_m^{(\beta)}(x),\\ &= \exp[V(x)]\sum_{m=0}^{2d-2} C_m^{(\beta)}(x)Z_{m,n} + \sum P_{n,m}^{(\beta)}\tilde{\psi}_m^{(\beta)}(x),\end{aligned} \tag{8.6}$$

where $V'(x) = \frac{\mathrm{d}}{\mathrm{d}x}V(x)$ and $V'(z) = \frac{\mathrm{d}}{\mathrm{d}z}V(z)$. Here, we have used $\exp[-V(z)][\{V'(x) - V'(z)\}/(x-z)] = \sum_{m=0}^{2d-2} C_m^{(\beta)}(x)\phi_m^{(\beta)}(z)$. Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\psi}_n^{(\beta)}(x) = \sum P_{n,m}^{(\beta)}\tilde{\psi}_m^{(\beta)}(x), \qquad n \geqslant 2d. \tag{8.7}$$

Using equations (8.5) and (8.7), we get

$$\begin{aligned}&\tilde{\phi}_n^{\prime(4)}(x) = \tilde{\psi}_n^{(4)}(x) = \sum P_{n,m}^{(4)}\tilde{\phi}_m^{(4)}(x), \qquad \tilde{\psi}_n^{\prime(1)}(x) = 2\tilde{\phi}_n^{(1)}(x),\\ &\tilde{\phi}_n^{\prime(1)}(x) = \sum P_{n,m}^{(1)}\tilde{\phi}_m^{(1)}(x), \qquad n\geqslant 2d.\end{aligned} \tag{8.8}$$

Thus, for $n \geqslant 2d-1$, our chosen functions satisfy the same system of differential equations as done by the skew-orthogonal vectors $\phi_n^{(\beta)}(x)$ and $\psi_n^{(\beta)}(x)$ (2.1).

Having obtained the relations between $\tilde{\phi}^{(\beta)}(x)$ and $\tilde{\psi}^{(\beta)}(x)$, we will now show that these functions also satisfy the same recursion relations. We start with

$$\begin{aligned}x\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\psi}_n^{(\beta)}(x) &= x\frac{\mathrm{d}}{\mathrm{d}x}\exp[V(x)]\int_{\mathbb{R}}\frac{\psi_n^{(\beta)}(z)\exp[-V(z)]}{(x-z)}\mathrm{d}z,\\ &= xV'(x)\tilde{\psi}_n^{(\beta)}(x) - \exp[V(x)]\int_{\mathbb{R}}\frac{xV'(z)}{x-z}\psi_n^{(\beta)}(z)\exp[-V(z)]\,\mathrm{d}z\\ &\quad + \exp[V(x)]\int_{\mathbb{R}}\frac{x\psi_n^{(\beta)\prime}(z)}{x-z}\exp[-V(z)]\,\mathrm{d}z.\end{aligned}$$

Now, replacing $x\to x-z+z$ on the right-hand side, we get

$$\begin{aligned}x\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\psi}_n^{(\beta)}(x) &= \exp[V(x)]\int_{\mathbb{R}}\frac{xV'(x)-zV'(z)}{x-z}\psi_n^{(\beta)}(z)\exp[-V(z)]\,\mathrm{d}z\\ &\quad + \exp[V(x)]\int_{\mathbb{R}}\frac{\mathrm{d}}{\mathrm{d}z}\left(\psi_n^{(\beta)}(z)\exp[-V(z)]\right)\mathrm{d}z\\ &\quad + \exp[V(x)]\int_{\mathbb{R}}\frac{\exp[-V(z)]}{x-z}\left[z\frac{\mathrm{d}}{\mathrm{d}z}\left(\psi_n^{(\beta)}(z)\right)\right]\mathrm{d}z.\end{aligned} \tag{8.9}$$

Thus, we have

$$x\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\psi}_n^{(\beta)}(x) = \exp[V(x)]\sum_{m=0}^{2d-1} d_m^{(\beta)}(x)Z_{m,n} + \sum R_{nm}^{(\beta)}\tilde{\psi}_n^{(\beta)}(x), \qquad \beta = 1,$$
$$= \exp[V(x)]\sum_{m=0}^{2d-1} d_m^{(\beta)}(x)Z_{m,n} + \sum \left(R_{nm}^{(\beta)} - \delta_{n,m}\right)\tilde{\psi}_n^{(\beta)}(x), \qquad \beta = 4. \quad (8.10)$$

Here, we have used $\exp[-V(z)][\{xV'(x) - zV'(z)\}/(x-z)] = \sum_{m=0}^{2d-1} d_m^{(\beta)}(x)\phi_m^{(\beta)}(z)$. Thus, for $n \geqslant 2d$, we can see that the first term drops out and hence $\tilde{\psi}_n^{(\beta)}(x)$ satisfy equations (2.2), (2.3), respectively.

Similarly, for $\beta = 4$, we get for $n \geqslant 2d$

$$x\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\phi}_n^{(4)}(x) = x\tilde{\psi}_n^{(4)}(x),$$
$$= \exp[V(x)]\int_{\mathbb{R}} \frac{(x - z + z)}{x - z}\psi_n^{(4)}(z)\exp[-V(z)]\,\mathrm{d}z,$$
$$= \sum R_{n,m}^{(4)}\tilde{\phi}_m^{(4)}(x), \quad (8.11)$$

where the first term drops off due to the condition $n \geqslant 2d$, $d \geqslant 1$. For $\beta = 1$, we differentiate (8.10) to get

$$x\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\phi}_n^{(1)}(x) = \sum \left(R_{nm}^{(1)} - \delta_{n,m}\right)\tilde{\phi}_n^{(1)}(x), \qquad n \geqslant 2d. \quad (8.12)$$

This in turn proves that for $n \geqslant 2d$ the Cauchy-like transforms satisfy the same set of recursion relations (4.1) and (4.2) as done by the skew-orthogonal vectors.

Thus, we prove that our function is compatible under a system of difference-differential equation, i.e. the solution of the system of differential equation is also a simultaneous solution of the system of difference equation. We will now show that these solutions are also compatible with the system of deformation relations, i.e., they are also solutions of the system of PDEs.

We can see that

$$\frac{\partial}{\partial u_K}\tilde{\psi}_n^{(\beta)}(x) = \frac{\exp[V(x)]}{K}\int_{\mathbb{R}} \frac{x^K - z^K}{x - z}\psi_n^{(\beta)}(z)\exp[-V(z)]\,\mathrm{d}z + \sum \left(U_K^{\beta}\right)_{nm}\tilde{\psi}_m^{(\beta)}(x),$$
$$= \exp[V(x)]\sum_{m=0}^{k-1} a_m^{(\beta)}(x)Z_{m,n} + \sum \left(U_K^{\beta}\right)_{n,m}\tilde{\psi}_m^{(\beta)}(x), \quad (8.13)$$

where $\exp[-V(z)][\{x^K - z^K\}/(x-z)] = \sum_{m=0}^{K-1} a_m^{(\beta)}(x)\phi_m^{(\beta)}(z)$.

Thus, for $n \geqslant 2d$, $\tilde{\psi}_n^{(\beta)}(x)$ satisfy equation (3.2). Also, combining equations (8.8), (8.13), we can see that

$$\tilde{\phi}_n^{(\beta)}(x) = \sum \left(U_K^{\beta}\right)_{n,m}\tilde{\phi}_m^{(\beta)}(x), \qquad n \geqslant 2d. \quad (8.14)$$

Thus, for $n \geqslant 2d$, the new-functions $\tilde{\phi}_n^{(\beta)}(x)$ and $\tilde{\psi}_n^{(\beta)}(x)$ satisfy the system of differential-difference-deformation equation and hence are mutually compatible.

8.2. The other solutions

In this subsection, we will show that $\Psi_n^{(j,\beta)}(x)$ and $\Phi_n^{(j,\beta)}(x)$, $j = 0, \ldots, 2d-2$, are also solutions to the system of difference-deformation-differential equations. For this, we will characterize the skew-orthogonal functions $\phi_k^{(1)}(x)$ and $\psi_k^{(4)}(x)$ (which are polynomials of degrees k and $k + 2d - 1$, respectively) through a set of skew-orthogonality relations with respect to the functions $f_j(x)$. The expressions for these auxiliary solutions were suggested (for the case $\beta = 1$) by the common referee of this paper and [39] using expressions contained therein and extended by the present author to the case $\beta = 4$.

We will start with the identity for a polynomial $\pi_j(x)$ of order j as

$$\pi_j(x) = \exp[V(x)]\frac{\mathrm{d}}{\mathrm{d}x}(x^{j-2d+1}\exp[-V(x)]). \tag{8.15}$$

There are $2k+1$ orthogonality properties for the skew-orthogonal polynomials of orders $2k$ and $2k+1$. Starting with

$$Z_{j+2d-1,2k} = \int_{\mathbb{R}} \psi_{2k}^{(\beta)}\pi_{j+2d-1}\exp[-V(x)]\,\mathrm{d}x = 0, \qquad \forall j = 0,\ldots,2k-2d+1, \tag{8.16}$$

and using equation (8.15), we get the $2k-2d+2$ conditions for

$$\begin{aligned}&\beta = 1, \qquad Z_{2k,j+2d-1} = 2\int_{\mathbb{R}} \phi_{2k}^{(1)}(x)x^j\exp[-V(x)]\,\mathrm{d}x = 0,\\ &\forall j = 0,\ldots,2k-2d+1,\end{aligned} \tag{8.17}$$

and

$$\begin{aligned}&\beta = 4, \qquad Z_{2k,j+2d-1} = \int_{\mathbb{R}} \psi_{2k}^{'(4)}(x)x^j\exp[-V(x)]\,\mathrm{d}x = 0,\\ &\forall j = 0,\ldots,2k-2d+1,\end{aligned} \tag{8.18}$$

while for the remaining $2d-1$ conditions one can have

$$Z_{2k,j} = \int_{\mathbb{R}}\int_{\mathbb{R}} \phi_{2k}^{(1)}(x)y^j\epsilon(x-y)\exp[-V(y)]\,\mathrm{d}x\,\mathrm{d}y = 0, \qquad j = 0,\ldots,2d-2, \tag{8.19}$$

$$Z_{j,2k} = \int_{\mathbb{R}}\int_{\mathbb{R}} \psi_{2k}^{(4)}(x)y^{j+2d-1}\epsilon(x-y)\exp[-V(y)]\,\mathrm{d}x\,\mathrm{d}y = 0, \qquad j = 0,\ldots,2d-2. \tag{8.20}$$

Thus, we define the function $f_j(x)$:

$$f_j(x) := \exp[V(x)]\int_{\mathbb{R}} \epsilon(x-y)y^j\exp[-V(y)]\,\mathrm{d}y, \qquad j = 0,\ldots,2d-2. \tag{8.21}$$

Remark. Here, we note that for $\beta = 4$, without loss of generality, $f_j(x)$ can also be defined as

$$f_j(x) := \exp[V(x)]\int y^j\exp[-V(y)]\,\mathrm{d}y + c, \qquad j = 0,\ldots,2d-2, \tag{8.22}$$

where c is a constant that can be fixed by skew-orthonormalization condition.

Having defined $f_j(x)$, we will show that the functions,

$$\begin{aligned}\Psi_n^{(j,\beta)}(x) &:= \int_{\mathbb{R}} \frac{f_j(x)-f_j(z)}{x-z}\Psi_n^{(\beta)}(z)\exp[-V(z)]\,\mathrm{d}z;\\ \Phi_n^{(j,\beta)}(x) &:= \int_{\mathbb{R}} \frac{f_j(x)-f_j(z)}{x-z}\Phi_n^{(\beta)}(z)\exp[-V(z)]\,\mathrm{d}z,\end{aligned} \tag{8.23}$$

satisfy the same system of difference-differential-deformation equations.

Componentwise, using equations (1.19) and (1.20), we get

$$\psi_n^{(j,4)}(x) = \sum P_{n,m}^{(4)}\phi_m^{(j,4)}(x), \qquad 2\phi_n^{(j,1)}(x) = \sum P_{n,m}^{(1)}\psi_m^{(j,1)}(x). \tag{8.24}$$

We also get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\psi_n^{(j,\beta)}(x) &= f_j(x)\int_{\mathbb{R}}\frac{V'(x)-V'(z)}{x-z}\psi_n^{(\beta)}(z)\exp[-V(z)]\,\mathrm{d}z\\ &\quad+2\int_{\mathbb{R}}\frac{x^j-z^j}{x-z}\psi_n^{(\beta)}(z)\exp[-V(z)]\,\mathrm{d}z\\ &\quad+\int_{\mathbb{R}}\frac{f_j(x)-f_j(z)}{x-z}\psi'^{(\beta)}_n(z)\exp[-V(z)]\,\mathrm{d}z\\ &= f_j(x)\sum_{m=0}^{2d-2}C_m^{(\beta)}(x)Z_{m,n}+2\sum_{m=0}^{j-1}a_m^{(\beta)}(x)Z_{m,n}+\sum P_{n,m}^{(\beta)}\psi_m^{(j,\beta)}(x).\end{aligned} \tag{8.25}$$

Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\psi_n^{(j,\beta)}(x)=\sum P_{n,m}^{(\beta)}\psi_m^{(j,\beta)}(x),\qquad n\geqslant 2d. \tag{8.26}$$

Using equations (8.24) and (8.26), we get

$$\begin{aligned}&\phi'^{(j,4)}_n(x)=\psi_n^{(j,4)}(x)=\sum P_{n,m}^{(4)}\phi_m^{(j,4)}(x),\qquad \psi'^{(j,1)}_n(x)=2\phi_n^{(j,1)}(x),\\ &\phi'^{(j,1)}_n(x)=\sum P_{n,m}^{(1)}\phi_m^{(j,1)}(x),\qquad n\geqslant 2d.\end{aligned} \tag{8.27}$$

Thus for $n\geqslant 2d$, our chosen functions satisfy the same system of differential equations as done by the skew-orthogonal vectors $\phi_n^{(\beta)}(x)$ and $\psi_n^{(\beta)}(x)$ (2.1).

Having obtained the relations between $\phi^{(j,\beta)}(x)$ and $\psi^{(j,\beta)}(x)$, we will now show that these functions also satisfy the same recursion relations. We start with

$$\begin{aligned}x\frac{\mathrm{d}}{\mathrm{d}x}\psi_n^{(j,\beta)}(x) &= f_j(x)\int_{\mathbb{R}}\frac{(xV'(x)-zV'(z))}{(x-z)}\psi_n^{(\beta)}(z)\exp[-V(z)]\,\mathrm{d}z\\ &\quad+2\int_{\mathbb{R}}\frac{(x^{j+1}-z^{j+1})}{(x-z)}\psi_n^{(\beta)}(z)\exp[-V(z)]\,\mathrm{d}z\\ &\quad+\int_{\mathbb{R}}\frac{\mathrm{d}}{\mathrm{d}z}\left(\psi_n^{(\beta)}(z)(f_j(x)-f_j(z))\exp[-V(z)]\right)\,\mathrm{d}z\\ &\quad+\int_{\mathbb{R}}z\psi'^{(\beta)}_n(z)\frac{f_j(x)-f_j(z)}{x-z}\exp[-V(z)]\,\mathrm{d}z,\end{aligned} \tag{8.28}$$

which for $\beta=1$ gives

$$\begin{aligned}&x\frac{\mathrm{d}}{\mathrm{d}x}\psi_n^{(j,1)}(x)=f_j(x)\sum_{m=0}^{2d-1}d_m^{(1)}(x)Z_{m,n}+2\sum_{m=0}^{j}a_m^{(1)}(x)Z_{m,n}+\sum R_{nm}^{(1)}\psi_n^{(j,1)}(x),\\ &j=0,\ldots,2d-2,\end{aligned} \tag{8.29}$$

and for $\beta=4$ gives

$$\begin{aligned}&x\frac{\mathrm{d}}{\mathrm{d}x}\psi_n^{(j,4)}(x)=f_j(x)\sum_{m=0}^{2d-1}d_m^{(4)}(x)Z_{m,n}+2\sum_{m=0}^{j}a_m^{(4)}(x)Z_{m,n}+\sum\left(R_{nm}^{(4)}-\delta_{n,m}\right)\psi_n^{(j,4)}(x),\\ &j=0,\ldots,2d-2.\end{aligned} \tag{8.30}$$

Thus, for $n\geqslant 2d$, we can see that the first two terms drop out and hence $\psi_n^{(j,\beta)}(x)$, like $\psi_n^{(B)}(x)$, satisfy equations (2.2) and (2.3).

Similarly, for $\beta = 4$, we get for $n \geqslant 2d$

$$\begin{aligned} x\frac{\mathrm{d}}{\mathrm{d}x}\phi_n^{(j,4)}(x) &= x\psi_n^{(j,4)}(x), \\ &= \sum R_{n,m}^{(4)}\phi_m^{(j,4)}(x), \end{aligned} \tag{8.31}$$

where the first term drops off due to the condition $n \geqslant 2d$, $d \geqslant 1$. For $\beta = 1$, we differentiate (8.10) to get

$$x\frac{\mathrm{d}}{\mathrm{d}x}\phi_n^{(j,1)}(x) = \sum \left(R_{nm}^{(1)} - \delta_{n,m}\right)\phi_n^{(j,1)}(x), \qquad n \geqslant 2d. \tag{8.32}$$

This in turn proves that for $n \geqslant 2d$ these functions satisfy the same set of recursion relations (4.1) and (4.2) as done by the skew-orthogonal vectors.

Thus, we prove that our function is compatible under a system of difference-differential equation, i.e. the solution of the system of differential equation is also a simultaneous solution of the system of difference equation. We will now show that these solutions are also compatible with the system of deformation relations i.e. they are also solutions of the system of PDEs.

We can see that

$$\begin{aligned} \frac{\partial}{\partial u_K}\psi_n^{(j,\beta)}(x) &= \frac{f_j(x)}{K}\int_{\mathbb{R}}\frac{x^K - z^K}{x - z}\psi_n^{(\beta)}(z)\exp[-V(z)]\,\mathrm{d}z + \sum\left(U_K^{\beta}\right)_{nm}\psi_m^{(j,\beta)}(x), \\ &= f_j(x)\sum_{m=0}^{k-1}a_m^{(\beta)}(x)Z_{m,n} + \sum\left(U_K^{\beta}\right)_{n,m}\psi_m^{(j,\beta)}(x). \end{aligned} \tag{8.33}$$

Thus, for $n \geqslant 2d$, $\psi_n^{(j,\beta)}(x)$ satisfy equation (3.2). Also, combining equations (8.27), (8.33), we can see that

$$\phi_n^{(j,\beta)}(x) = \sum\left(U_K^{\beta}\right)_{n,m}\phi_m^{(j,\beta)}(x), \qquad n \geqslant 2d. \tag{8.34}$$

Remark. Here, we encounter terms of the form $f_{j+K}(x)$, $j = 0, \ldots, 2d-2$ and $K = 1, \ldots, 2d$. For $(j+K) > 2d-2$, this term does not contribute, by definition. Also if we consider only $K = 2d$, i.e. the leading term in $V(x)$, no such problem arises.

Thus, for $n \geqslant 2d$, the new functions $\phi_n^{(j,\beta)}(x)$ and $\psi_n^{(j,\beta)}(x)$ satisfy the system of differential-difference-deformation equation and hence are mutually compatible.

Explicitly, we have

$$\begin{aligned} \Psi_{[W]}^{(\beta)} &= \begin{pmatrix} \Psi_{N-d}^{(\beta)}(x) & \tilde{\Psi}_{N-d}^{(\beta)}(x) & \Psi_{N-d}^{(0,\beta)}(x) & \cdots & \Psi_{N-d}^{(2d-2,\beta)}(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{N+d-1}^{(\beta)}(x) & \tilde{\Psi}_{N+d-1}^{(\beta)}(x) & \Psi_{N+d-1}^{(0,\beta)}(x) & \cdots & \Psi_{N+d-1}^{(2d-2,\beta)}(x) \end{pmatrix}, \\ \Phi_{[W]}^{(\beta)} &= \begin{pmatrix} \Phi_{N-d}^{(\beta)}(x) & \tilde{\Phi}_{N-d}^{(\beta)}(x) & \Phi_{N-d}^{(0,\beta)}(x) & \cdots & \Phi_{N-d}^{(2d-2,\beta)}(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{N+d-1}^{(\beta)}(x) & \tilde{\Phi}_{N+d-1}^{(\beta)}(x) & \Phi_{N+d-1}^{(0,\beta)}(x) & \cdots & \Phi_{N+d-1}^{(2d-2,\beta)}(x) \end{pmatrix}. \end{aligned} \tag{8.35}$$

Thus, we have obtained the fundamental solution to the system of differential-difference-deformation equation, satisfied by the skew-orthogonal vectors. From here, it should be possible to define the RHP for these skew-orthogonal vectors. It has been obtained in [39] for $\beta = 1$ and can be extended to $\beta = 4$.

9. Compatibility of the finite difference-differential-deformation systems

In the previous section, we have obtained the fundamental system of solutions for the differential-difference-deformation equation.

In this section, we will provide an alternate proof to show that the recursion relations, the linear differential equations and the deformation equations are compatible in the sense that they admit a basis of simultaneous solutions, provided the vectors form a certain algebra defined by the commutation relations (2.6) and (3.4). In other words, here, we will not use the explicit form of the fundamental solution to prove the compatibility.

This is the same as saying that the shifts $W \mapsto W+1$ in equations (4.3), (4.7), respectively, are compatible as vector differential-difference systems. This means that there exists a sequence of fundamental matrix solutions $\Phi^{(\beta)}_{[\mathbf{W}]}(x)$ and $\Psi^{(\beta)}_{[\mathbf{W}]}(x)$ which are simultaneous solutions of

$$\Phi^{(\mathbf{4})}_{[\mathbf{W+1}]}(x) = A^{(4)}_N(x)\Phi^{(\mathbf{4})}_{[\mathbf{W}]}(x), \tag{9.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi^{(\mathbf{4})}_{[\mathbf{W}]}(x) = D^{(4)}_N(x)\Phi^{(\mathbf{4})}_{[\mathbf{W}]}(x), \tag{9.2}$$

and

$$\Psi^{(\mathbf{4})}_{[\mathbf{W+1}]}(x) = B^{(4)}_N(x)\Psi^{(\mathbf{4})}_{[\mathbf{W}]}(x), \tag{9.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Psi^{(\mathbf{4})}_{[\mathbf{W}]}(x) = \underline{D}^{(4)}_N(x)\Psi^{(\mathbf{4})}_{[\mathbf{W}]}(x), \tag{9.4}$$

respectively. The same result holds for $\beta = 1$. This means that there exists a sequence of fundamental matrix solutions satisfying the recursion and differential equations

$$\Psi^{(\mathbf{1})}_{[\mathbf{W+1}]}(x) = A^{(1)}_N(x)\Psi^{(\mathbf{1})}_{[\mathbf{W}]}(x), \tag{9.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Psi^{(\mathbf{1})}_{[\mathbf{W}]}(x) = D^{(1)}_N(x)\Psi^{(\mathbf{1})}_{[\mathbf{W}]}(x), \tag{9.6}$$

and

$$\Phi^{(\mathbf{1})}_{[\mathbf{W+1}]}(x) = B^{(1)}_N(x)\Phi^{(\mathbf{1})}_{[\mathbf{W}]}(x), \tag{9.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi^{(\mathbf{1})}_{[\mathbf{W}]}(x) = \underline{D}^{(1)}_N(x)\Phi^{(\mathbf{1})}_{[\mathbf{W}]}(x), \tag{9.8}$$

respectively.

One must note that the differential equations are also compatible with the shift $W \mapsto W - 1$. The proof is exactly similar to the above case.

Similar procedure will be repeated to show the compatibility between the difference-deformation equation.

9.1. Proof: difference-differential equation

Now we will prove the compatibility of the differential-difference equation for $\Phi^{(4)}(x)$. The others are exactly similar and hence not repeated. The proof follows on similar line to the one outlined for bi-orthogonal polynomials [19].

Let us define a finite subsequence (or window) containing functions:

$$\overline{\Phi}^{(4)}_W(x) := \left[\overline{\Phi}^{(4)^t}_{N-d}(x), \ldots, \overline{\Phi}^{(4)^t}_{N+d-1}(x)\right]^t, \qquad N \geqslant d, \tag{9.9}$$

which is a solution to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\overline{\Phi}_W^{(4)}(x) = D_N^{(4)}(x)\overline{\Phi}_W^{(4)}(x). \tag{9.10}$$

A glance through the window will confirm that the shift $W \longmapsto W+1$ will introduce one new function $\overline{\Phi}_{N+d}^{(4)}(x)$ inside the shifted window. Componentwise, let us suppose that they satisfy the recursion relation

$$[\eta_{-d}(m) - x\zeta_{-d}(m)]\,\overline{\Phi}_{m+d}^{(4)}(x) = x\sum_{l=-d+1}^{d}\zeta_l(m)\overline{\Phi}_{m-l}^{(4)}(x) - \sum_{l=-d+1}^{d}\eta_l(m)\overline{\Phi}_{m-l}^{(4)}(x), \qquad m \geqslant d. \tag{9.11}$$

It is easy to see that the differential equation componentwise reads

$$\frac{\mathrm{d}}{\mathrm{d}x}\overline{\Phi}_n^{(4)}(x) = \sum_{l=-d}^{d}\zeta_j(n)\overline{\Phi}_{n-l}^{(4)}(x), \qquad n = N-d,\ldots,N+d-1, \tag{9.12}$$

where $\overline{\Phi}_m^{(4)}(x)$ outside the window can be expressed in terms of that within the window by using equation (9.11) recursively. To show compatibility between the shift and differential operator, we need to show that the newly defined function, which is assumed to satisfy

$$\overline{\Phi}_{N+d}^{(4)}(x) = (\eta_{-d}(N))^{-1}\left[x\sum_{l=-d}^{d}\zeta_l(N)\overline{\Phi}_{N-l}^{(4)}(x) - \sum_{l=-d+1}^{d}\eta_l(N)\overline{\Phi}_{N-l}^{(4)}(x)\right], \tag{9.13}$$

will also satisfy the same differential equation, i.e., we must show that the newly defined function $\overline{\Phi}_{N+d}^{(4)}(x)$ will also be a solution to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\overline{\Phi}_{N+d}^{(4)}(x) = \sum_{l=-d}^{d}\zeta_j(N+d)\overline{\Phi}_{N-l+d}^{(4)}(x). \tag{9.14}$$

Remark. One must be careful about the position of the inverse of the quaternion, as, in general, they do not commute.

Once this is proved, we can argue by induction that $\overline{\Phi}_{N+d+j}^{(4)}(x)$ satisfy the same sort of differential equation for any $j > 1$. To do this, we compute

$$\begin{aligned}
\eta_{-d}(N)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\overline{\Phi}_{N+d}^{(4)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x\sum_{l=-d}^{d}\zeta_l(N)\overline{\Phi}_{N-l}^{(4)}(x) - \sum_{l=-d+1}^{d}\eta_l(N)\overline{\Phi}_{N-l}^{(4)}(x)\right) \\
&= \sum_{l=-d}^{d}\zeta_l(N)\overline{\Phi}_{N-l}^{(4)}(x) + \sum_{l=-d}^{d}\sum_{j=-d}^{d}\zeta_l(N)\eta_j(N-l)\overline{\Phi}_{N-l-j}^{(4)}(x) \\
&\quad - \sum_{l=-d}^{d}\sum_{j=-d}^{d}\eta_l(N)\zeta_j(N-l)\overline{\Phi}_{N-l-j}^{(4)}(x) \\
&\quad + \eta_{-d}(N)\sum_{l=-d}^{d}\zeta_l(N+d)\overline{\Phi}_{N-l+d}^{(4)}(x).
\end{aligned} \tag{9.15}$$

From equation (9.15), we can see that in order to prove equation (9.14) we will have to show that

$$\sum_{l=-d}^{d} \zeta_l \overline{\Phi}_{N-l}^{(4)}(x) + \sum_{l=-d}^{d} \sum_{j=-d}^{d} \zeta_l(N)\eta_j(N-l)\overline{\Phi}_{N-l-j}^{(4)}(x)$$
$$- \sum_{l=-d}^{d} \sum_{j=-d}^{d} \eta_l(N)\zeta_j(N-l)\overline{\Phi}_{N-l-j}^{(4)}(x) = 0. \tag{9.16}$$

But this is nothing but a direct consequence of equation (2.18), in terms of its components. We can repeat the same procedure for the shift $W \longmapsto W-1$. Finally, one can extend this argument by induction to $\overline{\Phi}_{N+r}^{(4)}(x)$, thereby completing the proof.

We can repeat the same procedure to the sequences

$$\overline{\Psi}_W^{(1)}(x) := \left[\overline{\Psi}_{N-d}^{(1)^t}(x), \ldots, \overline{\Psi}_{N+d-1}^{(1)^t}(x)\right]^t, \qquad N \geqslant d, \tag{9.17}$$

and with some minor modifications to

$$\overline{\Phi}_W^{(1)}(x) := \left[\overline{\Phi}_{N-d}^{(1)^t}(x), \ldots, \overline{\Phi}_{N+d-1}^{(1)^t}(x)\right]^t, \qquad N \geqslant d, \tag{9.18}$$

and

$$\overline{\Psi}_W^{(4)}(x) := \left[\overline{\Psi}_{N-d}^{(4)^t}(x), \ldots, \overline{\Psi}_{N+d-1}^{(4)^t}(x)\right]^t, \qquad N \geqslant d, \tag{9.19}$$

to prove compatibility conditions between the difference-differential equations satisfied by the skew-orthonormal vectors.

9.2. *Proof: difference-deformation equation*

We show that these functions are also compatible with the deformation equations. We will briefly outline the proof. We start with the same finite subsequence (or window) containing functions

$$\overline{\Phi}_W^{(4)}(x) := \left[\overline{\Phi}_{N-d}^{(4)}(x), \ldots, \overline{\Phi}_{N+d-1}^{(4)}(x)\right]^t, \qquad N \geqslant d, \tag{9.20}$$

which, componentwise, will satisfy the PDE

$$\frac{\partial}{\partial u_K} \overline{\Phi}_n^{(4)}(x) = \sum_{j=0}^{K} U_j^K(n)\overline{\Phi}_{n-j}^{(4)}(x) \qquad n = N-d, \ldots, N+d-1. \tag{9.21}$$

Then under a shift $W \longmapsto W+1$, we get the new function $\overline{\Phi}_{N+d}^{(4)}(x)$ defined by

$$(\eta_{-d}(N))\overline{\Phi}_{N+d}^{(4)}(x) = x\left[\sum_{l=-d}^{d} \zeta_l(N)\overline{\Phi}_{N-d+1}(x) - \sum_{l=-d+1}^{d} \eta_l(N)\overline{\Phi}_{N-l}^{(4)}(x)\right], \tag{9.22}$$

which should satisfy

$$\frac{\partial}{\partial u_K} \overline{\Phi}_{N+d}^{(4)}(x) = \sum_{j=0}^{K} U_j^K(N+d)\overline{\Phi}_{N+d-j}^{(4)}(x). \tag{9.23}$$

To do this, we compute

$$\left(\frac{\partial}{\partial u_K}\right)(\eta_{-d}(N))\overline{\Phi}_{N+d}^{(4)}(x) = \frac{\partial}{\partial u_K}\left(x\left[\sum_{l=-d}^{d} \zeta_l(N)\overline{\Phi}_{N-l}^{(4)}(x) - \sum_{l=-d+1}^{d} \eta_l(N)\overline{\Phi}_{N-l}^{(4)}(x)\right]\right),$$
$$= x\sum_{l=-d}^{d} \zeta_l'(N)\overline{\Phi}_{N-l}^{(4)}(x) + x\sum_{l=-d}^{d}\sum_{j=0}^{K} \zeta_l(N)U_j^K(N-l)\overline{\Phi}_{N-j-l}^{(4)}(x)$$

$$
\begin{aligned}
&- \sum_{j=-d}^{d} \eta_l'(N)\overline{\Phi}^{(4)}_{N-l}(x) - \sum_{l=-d}^{d}\sum_{j=0}^{K} \eta_l(N)U_j^K(N-l)\overline{\Phi}^{(4)}_{N-j-l}(x) \\
&+ \eta_{-d}(N)\sum_{j=0}^{K} U_j^K(N+d)\overline{\Phi}^{(4)}_{N+d-j}(x) + \eta_{-d}'(N)\overline{\Phi}^{(4)}_{N+d}(x),
\end{aligned}
\tag{9.24}
$$

where $'$ denotes $\frac{\partial}{\partial u_K}$. Rearranging the coefficients in equation (9.14), we can see that basically we will have to show that

$$
\begin{aligned}
0 = x\sum_{l=-d}^{d} \zeta_l'(N)\overline{\Phi}^{(4)}_{N-l}(x) &+ x\sum_{l=-d}^{d}\sum_{j=0}^{K} \zeta_l(N)U_j^K(N-l)\overline{\Phi}^{(4)}_{N-j-l}(x) \\
&- \sum_{j=-d}^{d} \eta_l'(N)\overline{\Phi}^{(4)}_{N-l}(x) - \sum_{l=-d}^{d}\sum_{j=0}^{K} \eta_l(N)U_j^K(N-l)\overline{\Phi}^{(4)}_{N-j-l}(x).
\end{aligned}
\tag{9.25}
$$

But this is nothing but a direct consequence of the string equation (3.4), in terms of its components. Hence, we get equation (9.23).

We can repeat the same procedure for the shift $W \longmapsto W-1$. Finally, one can extend this argument by induction to $\overline{\Phi}^{(4)}_{N+r}(x)$, thereby completing the proof.

We can repeat the same procedure to the sequences $\overline{\Psi}^{(1)}_{W}(x)$ and with some minor modifications to $\overline{\Phi}^{(1)}_{W}(x)$ and $\overline{\Psi}^{(4)}_{W}(x)$ to prove compatibility conditions between the difference-deformation equations satisfied by the skew-orthonormal vectors.

10. Conclusion

In conclusion, we have obtained a system of differential-difference-deformation equation for a finite subsequence of skew-orthonormal vectors. We see that similar to the orthogonal and bi-orthogonal polynomials in ordinary space the Cauchy-like transforms of these skew-orthogonal vectors, of sufficiently high order (to be precise, $N \geqslant 2d$), also satisfy the same differential-difference-deformation equations in the quaternion space. We also derive an integral representation of these skew-orthogonal vectors in order to obtain the fundamental system of solutions for the overdetermined system of ODEs, difference and deformation equations.

On the other hand, from our already existing knowledge of skew-orthogonal polynomials, we know that the final results of the RH analysis will have some distinct differences from the orthogonal polynomials. For example, the zeros of orthogonal polynomials are real, which are in contrast to that of the skew-orthogonal polynomials [4]. To understand these properties, one needs to extend the already existing theory for orthogonal polynomials and matrix RHP [11] to skew-orthogonal polynomials and q-matrix RHP. We wish to come back to this in a later publication.

Finally, we would like to mention a result of Dyson and Mehta. They showed [40] that for the circular ensembles 'the probability distribution of a set of N alternate eigenvalues of a matrix in the orthogonal ensemble of order $2N$ is identical with the probability distribution of the set of all eigenvalues of a matrix in symplectic ensemble of order N'. It would be interesting to know if something similar exists for ensembles with polynomial potential and the role (if any) played by the duality relations between different skew-orthogonal vectors of orthogonal and symplectic ensembles.

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